Section 2.1 selected answers
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Exercises 1–7, 10, 15–21 odd, 31, 39.

2. Let \( g : \mathbb{R}^2 \to \mathbb{R} \) be given by \( g(x, y) = 2x^2 + 3y^2 - 7 \).

(a) Find the domain and range of \( g \).

Since \( g \) is always defined, its domain is all of \( \mathbb{R}^2 \). Since \( x^2 \) and \( y^2 \) are always each \( \geq 0 \), the minimum value of \( g \) is \(-7 \). Therefore, its range is the interval \([-7, \infty)\).

(b) Restrict the domain of \( g \) to make it one-to-one.

The problem is that \( g \) has the same values for many values of \((x, y)\). For instance, on the entire ellipse \( 2x^2 + 3y^2 = 7 \), \( g \) has the value 0. One solution, out of many possible solutions, is to make \( y = 0 \) and \( x \geq 0 \). When that restriction is made, \( g \) becomes one-to-one.

(c) Restrict the codomain of \( g \) to make it onto.

You can always restrict the codomain to the range to make it onto.

4. Find the domain and range of \( f(x, y) = \ln(x + y) \).

In order for \( \ln \) to be defined, \( x + y \) has to be positive. In other words the domain of \( f \) is the set of \((x, y)\) such that \( x + y > 0 \).

Within that domain \( x + y \) can be any positive number, so the range of \( f \) is just the range of \( \ln \), and the range of \( \ln \) is all of \( \mathbb{R} \). In order for \( g \) to be defined, there are two requirements. First, \( 4 - x^2 - y^2 - z^2 \) has to be \( \geq 0 \) so that its square root is defined. Second, \( \sqrt{4 - x^2 - y^2 - z^2} \) cannot be 0, since it’s in a denominator. Together, these conditions require \( x^2 + y^2 + z^2 < 4 \). If you want to write that in set notation, it looks like

\[
\{(x, y, z) \mid x^2 + y^2 + z^2 < 4\}.
\]

This set can be described geometrically as the open ball in \( \mathbb{R}^3 \) of radius 2 about the origin.

Next, to determine the range, note that the denominator is a positive number, but it can’t be larger than 2 since the maximum of \( \sqrt{4 - x^2 - y^2 - z^2} \) is \( \sqrt{2} \). Since the denominator is between 0 and 2, its reciprocal is between \( \frac{1}{2} \) and \( \infty \). Thus, the range is the half-open interval \([\frac{1}{2}, \infty)\).

10. Let \( f : \mathbb{R}^3 \to \mathbb{R} \) be defined by \( f(x) = x + 3j \).

Write out the component functions of \( f \) in terms of the components of the vector \( x \).

Let \( x = (x, y, z) \) as usual. Then

\[
f(x) = x + 3j = (x, y, z) + 3(0, 1, 0) = (x, y + 3, z).
\]

Therefore, the component functions are \( f_1(x, y, z) = x, f_2(x, y, z) = y + 3, \) and \( f_3(x, y, z) = z \).

19. Determine level curves and sketch the graph of \( f(x, y) = xy \).

The curve of height \( c \) is the solution set for \( xy = c \). If \( c \neq 0 \), then the curve is a rectangular hyperbola whose asymptotes are the \( x \)- and \( y \)-axes. When \( c = 0 \), the level curve is the union of the \( x \)- and \( y \)-axes. The surface is called a hyperbolic paraboloid.

31. Given a function \( f(x, y) \), can two different level curves of \( f \) intersect? Why or why not?

If the level curve for height \( c \), which is \( f(x, y) = c \), intersects the level curve for height \( d \), which is \( f(x, y) = d \), then \( c = d \). Therefore, the level curves for two distinct heights cannot intersect.
39. Is it possible to find a function \( f(x, y) \) so that the ellipsoid \( \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \) is the graph \( z = f(x, y) \)?

To be the graph of a function \( f(x, y) \), for a given vector \((x, y)\) there can be at most one value \( z \) such that \((x, y, z)\) lies on the surface. But for this surface, there will be two, both \( z \) and \(-z\). So this surface is not the graph of a function. It is, however, the union of the graphs of two functions

\[
z = \pm \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}
\]