Section 4.2 selected answers
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Exercises from section 4.2: 3–6, 13–16.

3. Determine the nature of the critical points of the function

\[ f(x, y) = 2xy - 2x^2 - 5y^2 + 4y - 3. \]

Note that this is a quadratic function, so its graph is one of the quadric surfaces. In fact, it’s an elliptic paraboloid.

First find the critical points by seeing where the two partial derivatives are simultaneously 0. The partial derivatives are

\[ f_x = 2y - 4x \]
\[ f_y = 2x - 10y + 4 \]

They’re both 0 only at \( a = (\frac{2}{9}, \frac{4}{9}) \). So \( a \) is the only critical point. To see what kind of critical point it is, look at the Hessian.

\[
Hf(a) = \begin{bmatrix} f_{xx}(a) & f_{xy}(a) \\ f_{yx}(a) & f_{yy}(a) \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -10 \end{bmatrix}
\]

Now evaluate the two principal minors \( d_1 \) and \( d_2 \). The first principal minor \( d_1 \) is just \(-4\), the upper left entry of the Hessian. The second principal minor \( d_2 \) is the determinant of the Hessian,

\[
d_2 = |Hf(a)| = \begin{vmatrix} -4 & 2 \\ 2 & -10 \end{vmatrix} = 36.
\]

Since all the odd principal minors (there’s only one of them), are negative, while all the even principal minors (again, there’s only one of them), are positive, therefore the second derivative test says this critical point is a local maximum.

4. Determine the nature of the critical points of the function

\[ f(x, y) = \ln(x^2 + y^2 + 1). \]

You don’t need the theory we developed in this section to see that \((0, 0)\) is the only critical point and it’s a minimum. Since \( x^2 + y^2 + 1 \) is a paraboloid, and \( \ln \) is an increasing function, the graph of this \( f \) looks like a squashed paraboloid. But since the purpose of this exercise is to become better acquainted with the second derivative test, let’s use it.

Follow the steps outlined in the previous exercise.

\[ f_x = \frac{2x}{x^2 + y^2 + 1} \]
\[ f_y = \frac{2y}{x^2 + y^2 + 1} \]

These two first partial derivatives are 0 only at \( a = (0, 0) \), the only critical point.

\[
\begin{align*}
f_{xx} &= \frac{-2x^2 + 2y^2 + 2}{(x^2 + y^2 + 1)^2} \\
f_{xy} &= \frac{-4xy}{(x^2 + y^2 + 1)^2} \\
f_{yy} &= \frac{2x^2 - 2y^2 + 2}{(x^2 + y^2 + 1)^2}
\end{align*}
\]
\[ Hf(a) = \begin{bmatrix} f_{xx}(a) & f_{xy}(a) \\ f_{yx}(a) & f_{yy}(a) \end{bmatrix} \]
\[ = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \]
\[ d_1 = 2 \]
\[ d_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 \]

Since all the principal minors are positive, therefore, by the second derivative test, this critical point is a local minimum.

5. Determine the nature of the critical points of the function

\[ f(x, y) = x^2 + y^3 - 6xy + 3x + 6y. \]

Find the critical points.
\[ f_x = 2x - 6y + 3 \]
\[ f_y = 3y^2 - 6x + 6 \]

The two equations \(2x - 6y + 3 = 0\) and \(3y^2 - 6x + 6 = 0\) have two simultaneous solutions, namely, \((\frac{3}{2}, 1)\) and \((2\frac{7}{8}, 5)\), so they’re the two critical points of \(f\).

\[ f_{xx} = 2 \]
\[ f_{xy} = -6 \]
\[ f_{yy} = 6y \]

First, look at the critical point \((\frac{3}{2}, 1)\).

\[ Hf(\frac{3}{2}, 1) = \begin{bmatrix} 2 & -6 \\ -6 & 6 \end{bmatrix} \]
\[ d_1 = 2 \]
\[ d_2 = \begin{vmatrix} 2 & -6 \\ -6 & 6 \end{vmatrix} = -24 \]

This critical point \((\frac{3}{2}, 1)\) is neither a local minimum nor a local maximum, but \(d_2\) is not 0, so it’s a saddle point.

Next, look at the critical point \((2\frac{7}{8}, 5)\).

\[ Hf(2\frac{7}{8}, 5) = \begin{bmatrix} 2 & -6 \\ -6 & 30 \end{bmatrix} \]
\[ d_1 = 2 \]
\[ d_2 = \begin{vmatrix} 2 & -6 \\ -6 & 30 \end{vmatrix} = 24 \]

Since both principal minors are positive, this critical point \((2\frac{7}{8}, 5)\) is a minimum.

6. Determine the nature of the critical points of the function

\[ f(x, y) = y^4 - 2xy^2 + x^3 - x. \]

Find the first partial derivatives.
\[ f_x = -2y^2 + 3x^2 - 1 \]
\[ f_y = 4y^3 - 4xy \]

Next, solve the two equations \(-2y^2 + 3x^2 - 1 = 0\) and \(4y^3 - 4xy = 0\) simultaneously. After dividing the second one by 4 and factoring, you get \(y(y^2 - x) = 0\), so there are two cases: either \(y = 0\) or \(y^2 - x = 0\). In the case when \(y = 0\), the first equation has two solutions, namely \(x = \pm \sqrt{1/3}\). In the other case when \(y^2 - x = 0\), the first equation simplifies to \(3x^2 - 2x - 1 = 0\), and that has two solutions, namely \(x = 1\), \(-\frac{1}{3}\). Now, when \(x = 1\), the equation \(y^2 - x = 0\) has two solutions, \(y = \pm 1\), but when \(x = -\frac{1}{3}\), the equation \(y^2 - x = 0\) has no solutions. Thus, there are four critical points:

\((0, \sqrt{1/3}), (0, -\sqrt{1/3}), (1, 1), (1, -1)\).

Next, find the second partial derivatives.
\[ f_{xx} = 6x \]
\[ f_{xy} = -4y \]
\[ f_{yy} = 12y^2 - 4x \]
The general Hessian matrix is

\[
Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}
\]

\[
= \begin{bmatrix} 6x & -4y \\ -4y & 12y^2 - 4x \end{bmatrix}
\]

So the general principal minors are

\[
d_1 = 6x
\]

\[
d_2 = \begin{vmatrix} 6x & -4y \\ -4y & 12y^2 - 4x \end{vmatrix}
= 72xy^2 - 24x^2 - 16y^2
\]

To determine the nature of the four critical points, these two principal minors need to be evaluated at the four points.

<table>
<thead>
<tr>
<th>a</th>
<th>(d_1(a))</th>
<th>(d_2(a))</th>
<th>conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0, \sqrt{1/3}))</td>
<td>0</td>
<td>-16/9</td>
<td>saddle point</td>
</tr>
<tr>
<td>((0, -\sqrt{1/3}))</td>
<td>0</td>
<td>-16/9</td>
<td>saddle point</td>
</tr>
<tr>
<td>((1, 1))</td>
<td>6</td>
<td>32</td>
<td>local min</td>
</tr>
<tr>
<td>((1, -1))</td>
<td>6</td>
<td>32</td>
<td>local min</td>
</tr>
</tbody>
</table>

For \(f_x\) to be 0, we need \(x = -\frac{1}{2}\). For \(f_y\) to be 0, we need \(y = \frac{1}{3}\). So the only critical point is \(a = (-\frac{1}{2}, \frac{1}{3})\).

\[
f_x = 2 + 1/x \\
f_y = -3 + 1/y
\]

At \(a\) these three second partials are -4, 0, and -9, respectively. Therefore,

\[
Hf(a) = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}
\]

\[
d_1 = -4 \\
d_2 = 36
\]

Since \(d_1\) is negative while \(d_2\) is positive, therefore, by the second derivative test, this critical point is a local maximum.

14. Identify and determine the nature of the critical points of the function \(f(x, y) = \cos x \sin y\).

If you want, you can examine the function a little first to figure out what’s going on. If \(y\) is an integer times \(\pi\), then \(z = f(x, y) = 0\); also if \(x\) is an integer plus \(\frac{1}{2}\) times \(\pi\), then \(z\) is 0. So the graph of this function meets the \((x, y)\)-plane in two sets of parallel lines that divide the \((x, y)\)-plane into squares. Inside each square, \(z\) is either always positive, or \(z\) always negative. Right in the middle of each square, there will be an extremum of \(f\).
Find the first partial derivatives.

\[ f_x = -\sin x \sin y \]
\[ f_y = \cos x \cos y \]

Solve the pair of equations \(- \sin x \sin y = 0\) and \(\cos x \cos y = 0\) to determine the critical points. If the product \(\sin x \sin y\) is 0, then either \(\sin x\) or \(\sin y\) is 0, so either \(x\) is an integer times \(\pi\) or \(y\) is an integer times \(\pi\). Likewise, since the product \(\cos x \cos y\) is 0, either \(x\) is an integer plus \(\frac{1}{2}\) times \(\pi\), or \(y\) is an integer plus \(\frac{1}{2}\) times \(\pi\). Putting these conditions together, we get two cases.

Case 1: \(x\) is an integer times \(\pi\) while \(y\) is an integer plus \(\frac{1}{2}\) times \(\pi\), that is, \((x, y) = (m\pi, (n + \frac{1}{2})\pi)\). These case 1 points are at the centers of the squares mentioned above.

Case 2: \(y\) is an integer times \(\pi\) while \(x\) is an integer plus \(\frac{1}{2}\) times \(\pi\), that is, \((x, y) = ((m + \frac{1}{2})\pi, n\pi)\). These case 2 points are at the corners of the squares mentioned above.

Next, find the second partial derivatives.

\[ f_{xx} = -\cos x \sin y \]
\[ f_{xy} = -\sin x \cos y \]
\[ f_{yy} = -\cos x \sin y \]

The general Hessian matrix is

\[
Hf = \begin{bmatrix}
  f_{xx} & f_{xy} \\
  f_{yx} & f_{yy}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  -\cos x \sin y & -\sin x \cos y \\
  -\sin x \cos y & -\cos x \sin y
\end{bmatrix}
\]

So the general principal minors are

\[ d_1 = -\cos x \sin y \]
\[ d_2 = \begin{vmatrix}
  -\cos x \sin y & -\sin x \cos y \\
  -\sin x \cos y & -\cos x \sin y
\end{vmatrix}
\]

\[ = \cos^2 x \sin^2 y - \sin^2 x \cos^2 y \]

Next, evaluate the principal minors for the critical points.

Case 1 critical points: \((x, y) = (m\pi, (n + \frac{1}{2})\pi)\).

For all these critical points, \(\sin x = 0\) and \(\cos x = \pm 1\), while \(\sin y = \pm 1\) and \(\cos y = 0\). Therefore \(d_1\) is either +1 or −1, but \(d_2 = 1\). So, if \(d_1 = -1\), then the critical point is a max, but if \(d_1 = +1\), then the critical point is a min.

Case 2 critical points: \((x, y) = ((m + \frac{1}{2})\pi, n\pi)\).

For all these critical points, \(\sin x = \pm 1\) and \(\cos x = 0\), while \(\sin y = 0\) and \(\cos y = \pm 1\). Therefore, \(d_1 = 0\) and \(d_2 = -1\). Hence, all these critical points are saddle points.

16. Identify and determine the nature of the critical points of the function

\[ f(x, y, z) = (x^2 + 2y^2 + 1) \cos z. \]

Find the first partial derivatives.

\[ f_x = 2x \cos z \]
\[ f_y = 4y \cos z \]
\[ f_z = -(x^2 + 2y^2 + 1) \sin z \]

Set these all to 0 and solve to find the critical points. Note that \(-(x^2 + 2y^2 + 1) \sin z = 0\) implies \(\sin z = 0\) since \(x^2 + 2y^2 + 1\) can’t be 0. Therefore \(z\) is an integral multiple of \(\pi\). But then \(\cos z\) is ±1, and the other two equations, \(2x \cos z = 0\) and \(4y \cos z = 0\), imply both \(x\) and \(y\) equal 0. Thus, the
critical points are of the form \((x, y, z) = (0, 0, n\pi)\) where \(n\) is an integer.

Next, find the second partial derivatives.

\[
\begin{align*}
f_{xx} &= 2 \cos z \\
f_{xy} &= 0 \\
f_{xz} &= -2x \sin z \\
f_{yy} &= 4 \cos z \\
f_{yz} &= -4y \sin z \\
f_{zz} &= -(x^2 + 2y^2 + 1) \cos z
\end{align*}
\]

The general Hessian matrix is

\[
Hf = \begin{bmatrix}
  f_{xx} & f_{xy} & f_{xz} \\
  f_{yx} & f_{yy} & f_{yz} \\
  f_{zx} & f_{zy} & f_{zz}
\end{bmatrix}
= \begin{bmatrix}
  2 \cos z & 0 & -2x \sin z \\
  0 & 4 \cos z & -4y \sin z \\
-2x \sin z & -4y \sin z & -(x^2 + 2y^2 + 1) \cos z
\end{bmatrix}
\]

So the general principal minors are

\[
\begin{align*}
d_1 &= 2 \cos z \\
d_2 &= \begin{vmatrix} 2 \cos z & 0 \\ 0 & 4 \cos z \end{vmatrix} = 8 \cos^2 z \\
d_3 &= \begin{vmatrix} 2 \cos z & 0 & -2x \sin z \\ 0 & 4 \cos z & -4y \sin z \\ -2x \sin z & -4y \sin z & -(x^2 + 2y^2 + 1) \cos z \end{vmatrix}
\end{align*}
\]

Now evaluate these principal minors at the critical point \((x, y, z) = (0, 0, n\pi)\).

\[
\begin{align*}
d_1 &= 2 \cos n\pi \\
d_2 &= 8 \\
d_3 &= \begin{vmatrix} 2 \cos n\pi & 0 & 0 \\ 0 & 4 \cos n\pi & 0 \\ 0 & 0 & -\cos n\pi \end{vmatrix} \\
&= -8 \cos^3 n\pi
\end{align*}
\]

There are two cases depending on whether \(n\) is even or \(n\) is odd. If \(n\) is even, then \(\cos n\pi = 1\), so \(d_1\) and \(d_2\) are positive, but \(d_3\) is negative, so the critical point is a saddle point. But if \(n\) is odd, then \(\cos n\pi = -1\), so \(d_1\) is negative, but \(d_2\) and \(d_3\) are positive, and again the critical point is a saddle point. Thus, all the critical points are saddle points.