CLARK UNIVERSITY 1887

Surfaces
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Preview. We'll introduce parametrized surfaces so we can study surface integrals and generalize the results we've seen from dimension 2 to dimension 3. In dimension 2, we had regions in the plane and their boundaries were closed curves. Then we took double integrals over the plane regions and line integrals over the curves. In dimension 3, we'll have regions in space and their boundaries will be surfaces. Then we'll take triple integrals over the solid regions and surface integrals over the surfaces.

That means we'll have to define surface integrals, and before that we'll look at surfaces in more detail.

Surfaces in $\mathbf{R}^3$. The general principle of analytic geometry says that a surface in $\mathbf{R}^3$ is given by an equation in three variables x, y, and z. When the equation is linear, then the surface is a plane, but when it's not linear, then some other surface is described.

Of course, graphs $f(x,y) = z$ of functions $f : \mathbf{R}^2 \to \mathbf{R}$ of two variables are surfaces over the (x,y)-plane. But there are other surfaces that aren't graphs of such functions like the unit sphere, which has the equation $x^2 + y^2 + z^2 = 1$.

Lots of surfaces arise as level sets of functions $f : \mathbf{R}^3 \to \mathbf{R}$ of three variables, the surface being given by the equation $f(x,y,z) = c$ for the level c. For example, the unit sphere is the $c = 1$ level set of the function $f(x,y,z) = x^2 + y^2 + z^2$.

Parameterizing surfaces. Those equations are fine for describing surfaces, but they don't work as well for doing calculus with surfaces. A more useful way to describe a surface is by parametrizing it. Whereas a line is parametrized by one variable—we've usually used t—a surface is parametrized by two variables—we'll usually use s and t.

It helps to have a standard notation. We'll use S to denote the surface in $\mathbf{R}^3$ that we're parametrizing. We'll use $\mathbf{X}(s,t)$ to describe the parametrization. Then $\mathbf{X}$ is a vector-valued function $\mathbf{R}^2 \to \mathbf{R}^3$. The domain D of $\mathbf{X}$ is usually not all of $\mathbf{R}^2$. The coordinate functions of $\mathbf{X}$ are denoted x, y, and z, so

$$\mathbf{X}(s,t) = (x(s,t), y(s,t), z(s,t)).$$

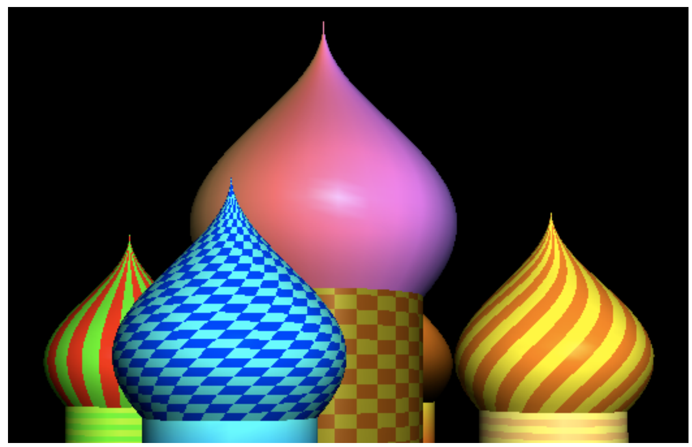

Figure 1: Onion domes

Example 1 (An onion dome)**.** Onion domes are domes often seen at the tops of Russian Orthodox churches such as the colorful domes on Saint Basil's Cathedral in Moscow. We can model one with the function $\mathbf{X}$ defined by

$$[x, y, z] = \begin{bmatrix} (1+\cos s)\cos t \\ (1+\sin s)\sin t \\ s \end{bmatrix}$$

where the domain D of X is the rectangle $[-\frac{\pi}{3}, \pi] \times [0, 2\pi]$. This surface S is a surface of revolution about the z. The variable t describes an angle around the z-axis. Since z is just s, the base of the dome is at height $z = -\frac{\pi}{3}$, and the tip is at $z = \pi$. The radius of the dome at height z is $1 + \cos z$. At the base, that radius is 1.5, increasing to 2 at $z = 0$, then decreasing to 0 at the tip.

For examples of more surfaces and their parameterizations see the Gallery of Surfaces at http://math.clarku.edu/~djoyce/ma131/gallery.pdf

The tangent plane at a point. Let's consider a particular point on the surface. Let it be the point $\mathbf{X}(s_0, t_0)$, and let's also denote it

$$\mathbf{X}(s_0, t_0) = (x_0, y_0, z_0) = (x(s_0, t_0), y(s_0, t_0), z(s_0, t_0)).$$

We can think of the parametrization $\mathbf{X}(s, t)$ as placing and (s, t)-coordinate system on the surface S. When t has a specific value t_0, we get a "latitude" on S, called an s-coordinate curve, and it's denoted $\mathbf{X}(s, t_0)$. Likewise, when s has a specific value s_0, we get a "longitude" on S, called a t-coordinate curve, and denoted $\mathbf{X}(s_0, t)$. These two coordinate curves pass through the point $\mathbf{X}(s_0, t_0)$.

The tangent vector $\mathbf{T}_s$ to the coordinate curve $\mathbf{X}(s, t_0)$ is

$$\mathbf{T}_s(s_0, t_0) = \frac{\partial \mathbf{X}}{\partial s} = \left(\frac{\partial x}{\partial s}(s_0, t_0), \frac{\partial y}{\partial s}(s_0, t_0), \frac{\partial z}{\partial s}(s_0, t_0) \right),$$

while the tangent vector $\mathbf{T}_t$ to the coordinate curve $\mathbf{X}(s_0, t)$ is

$$\mathbf{T}_t(s_0, t_0) = \frac{\partial \mathbf{X}}{\partial t} = \left(\frac{\partial x}{\partial t}(s_0, t_0), \frac{\partial y}{\partial t}(s_0, t_0), \frac{\partial z}{\partial t}(s_0, t_0) \right).$$

Together, these two tangent vectors $\mathbf{T}_s$ and $\mathbf{T}_t$ span a tangent plane at the point $\mathbf{X}(s_0, t_0)$, at least in the case that the parametrization $\mathbf{X}$ is a C^1 function. (If it's not C_1, then there might not be a tangent plane.) The easiest way to describe this plane is to use the normal vector $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$. Properly written out in full $\mathbf{N}$ is

$$\mathbf{N}(s_0, t_0) = \mathbf{T}_s(s_0, t_0) \times \mathbf{T}_t(s_0, t_0).$$

With this definition of the normal vector $\mathbf{N}$, the equation of the plane tangent to the surface S at the point $\mathbf{X}(s_0, t_0)$ is

$$\mathbf{N}(s_0, t_0) \cdot (\mathbf{x} - \mathbf{X}(s_0, t_0)) = 0,$$

where $\mathbf{x}$ is the variable point (x, y, z) on that plane. If $\mathbf{N}(s_0, t_0)$ has coordinates (A, B, C), then we can write that equation as

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

Math 131 Home Page at
http://math.clarku.edu/~djoyce/ma131/