CLARK UNIVERSITY 1887

Intro to vector fields
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Introduction of vector fields. We'll examine vector fields. Some of those will be gradient fields, that is, vector fields which are gradients of scalar functions, but many won't be. We'll also look at the flow lines of vector fields.

A *vector field* is a vector-valued function $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ from a vector space to itself. We can do more with a vector field than a general vector-valued function because when the domain and codomain are the same, a vector $\mathbf{x}$ and the vector-field value $\mathbf{F}(\mathbf{x})$ can interact in ways that can't happen when the domain and codomain aren't the same. We'll be using them a lot in the rest of the course.

We can draw plane vector fields by attaching small arrows at points in the plane to represent the vectors at those points.

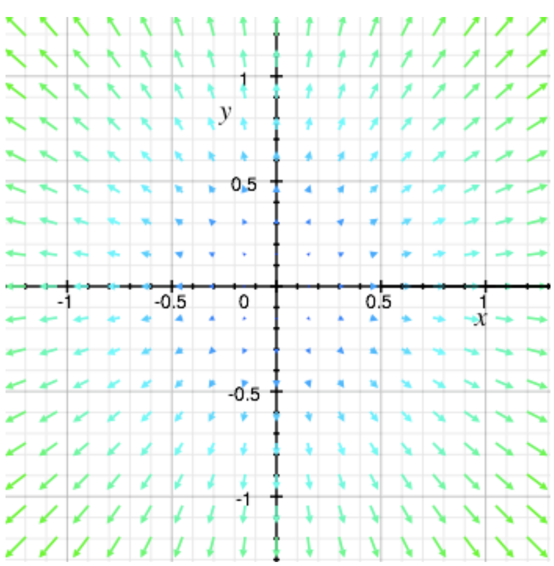

Figure 1: A radial vector field

Example 1 (A radial vector field)**.** We'll look as several vector fields in the plane, and draw them by drawing a few vectors $\mathbf{F}(\mathbf{x})$ with their tails attached to the points $\mathbf{x}$. For instance, the vector field $\mathbf{F}(\mathbf{x}) = (1, 1)$ is a constant vector field with all the vector field arrows being parallel.

Figure 1 illustrates an example of a radial field, namely the field $\mathbf{F}(x, y) = (x, y)$. All the vectors point away from the origin with short vectors nearer the origin and longer vectors far from the origin.

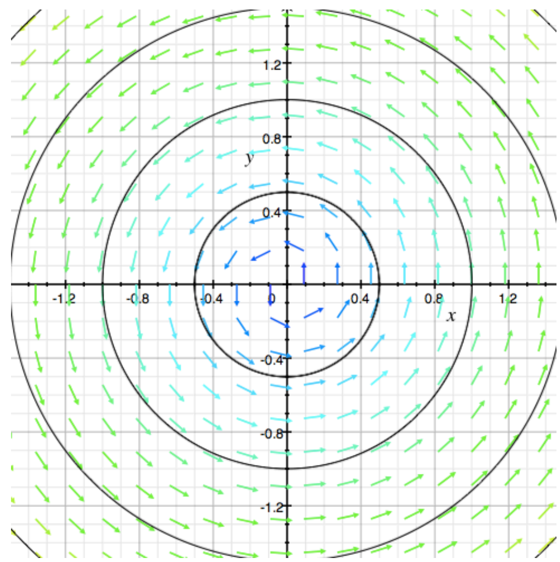

Figure 2: A circular vector field

Example 2 (A circular vector field)**.** Another example is the vector field $\mathbf{F}(x, y) = (-y, x)$ whose arrows rotate around the origin. It's shown in figure 2. Again, short vectors near the origin and long ones further away, but this time they don't point away from the origin but in a direction orthogonal to the direction to the origin.

Example 3 (Spatial vector field)**.** An example vector field on $\mathbf{R}^3$ is what is called the *inverse square vector field* defined by

$$\mathbf{F}(\mathbf{x}) = \frac{c}{\|\mathbf{x}\|^3}\,\mathbf{x}$$

where c is a constant. It's also a radial vector field. When $c = -GMm$, it describes the gravitational field with a point mass M at the origin and a point mass m at the point $\mathbf{r}$, where G is the universal gravitational constant. Alternatively, when $c = kq_1q_2$, it describes the electrostatic force between two point charges, q_1 and q_2, and k is a constant.

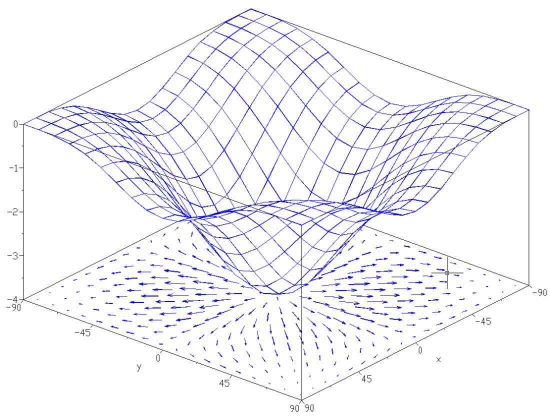

Figure 3: $z = f(x, y)$ and its gradient field

Gradient fields. Recall that when $f : \mathbf{R}^n \to \mathbf{R}$ is a scalar field, its gradient ∇f, which we will sometimes denote $\mathbf{F}$, is a vector field $\nabla f : \mathbf{R}^n \to \mathbf{R}^n$.

There's some special terminology that goes along with this. When a vector field $\mathbf{F}$ is the gradient of a scalar field f, we say $\mathbf{F}$ is a *gradient field*, and we say f is a *potential function* for $\mathbf{F}$. See figure 3 for an example potential function $f : \mathbf{R}^2 \to \mathbf{R}$ and its gradient field ∇f. Later, we'll find conditions on $\mathbf{F}$ that tell us when a vector-field is a gradient field.

Often, a potential function f for a vector field $\mathbf{F}$ is defined so that $\mathbf{F} = -\nabla f$ instead of $\mathbf{F} = +\nabla f$.

Example 4. A potential function for the inverse square field $\mathbf{F}(\mathbf{x}) = \dfrac{c}{\|\mathbf{x}\|^3}\mathbf{x}$ is the function $f(\mathbf{x}) = -\dfrac{c}{\|\mathbf{x}\|}$.

Equipotential sets. Suppose that $\mathbf{F}$ is the gradient field for the potential field f. Recall that the level set for f at a scalar constant c is the set

$$\{\mathbf{x} \mid f(\mathbf{x}) = c\}.$$

Level sets of potential fields are also called *equipotential sets*. When $n = 2$ they're called *equipotential lines (or curves)*, and when $n = 3$ they're called *equipotential surfaces*. Since, in general, gradients are orthogonal to level sets, therefore, the vectors of a vector field are orthogonal to potential sets. (See the diagrams in the text.)

Flow lines of vector fields. Let $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ be a vector field (not necessarily a gradient field). You can imagine some of the vectors in a vector field connected together to make a curve. More precisely, imagine a path $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ whose velocity vectors $\mathbf{x}'(t)$ are vectors in this vector field. Such a path is called a *flow line* of the vector field. The requirement that the velocity vectors be vectors in the vector field is the equation

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)).$$

Another way of saying this is that a flow line is a solution to a system of differential equation. When $n = 2$, the two differential equations are

$$\begin{aligned} x' &= F_1(x, y) \\ y' &= F_2(x, y) \end{aligned}$$

Example 5 (Constant vector field)**.** For the constant the vector field $\mathbf{F}(\mathbf{x}) = (1, 1)$, the flow lines are the parallel straight paths $(x(t), y(t)) = (t, t+c)$ where different values of the constant c give different lines.

Example 6 (Rotational vector field)**.** For the field $\mathbf{G}(x, y) = (-y, x)$ illustrated in figure 2, the flow lines are paths on circles of various radii c:

$$(x(t), y(t)) = (c \cos t, c \sin t)$$

since

$$(x'(t), y'(t)) = (-c \sin t, c \cos t) = (-y(t), x(t)).$$

Example 7 (Equiangular spirals)**.** Consider the vector field $\mathbf{T}(x, y) = (x - y, x + y)$ displayed in

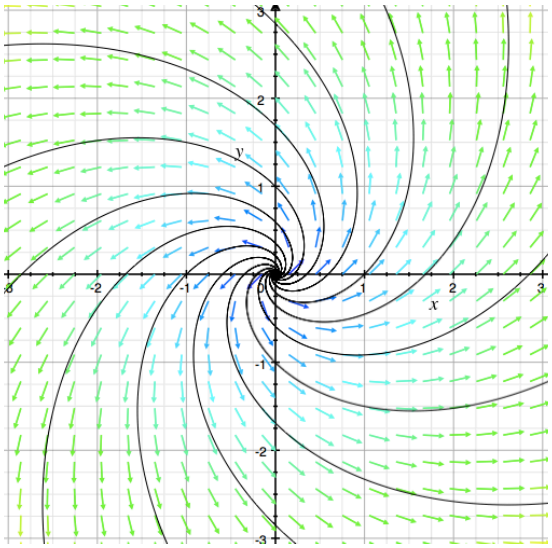

Figure 4: Equiangular spirals

figure 4. A flow line for this vector field satisfies the pair of differential equations

$$\begin{aligned} x' &= x - y \\ y' &= x + y \end{aligned}$$

The general solution to this differential equation is

$$(x, y) = (Ae^t \cos t, Ae^t \sin t)$$

where A is any constant. In polar form, the solutions are $r = Ae^t$, $\theta = t$. The curves $r = Ae^\theta$ are equiangular spirals, what Jacob Bernoulli (1654-1705) called *Spira mirabilis*.

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