Vector surface integrals
Math 131 Multivariate Calculus
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Summary of the surface differentials, surface areas, and scalar surface integrals that we already discussed.

The surface differential dS can be written in terms of of the normal vector $\mathbf{N}$, or the tangent vectors $\mathbf{T}_s$ and $\mathbf{T}_t$, or Jacobians as

$$\begin{aligned} dS &= \|\mathbf{N}\|\,ds\,dt \\ &= \|\mathbf{T}_s \times \mathbf{T}_t\|\,ds\,dt \\ &= \sqrt{\left(\tfrac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\tfrac{\partial(x,z)}{\partial(s,t)}\right)^2 + \left(\tfrac{\partial(x,y)}{\partial(s,t)}\right)^2}\,ds\,dt \end{aligned}$$

Any of these expressions of dS can be used to find the total area of the surface, which is the double integral $\displaystyle\iint_D dS$, where D is the domain of the parametrization $\mathbf{X}$ describing the surface.

The scalar surface integral for the scalar field f is just the double integral $\displaystyle\iint_D f\,dS$.

New topics. The definition of vector surface integrals, some examples, and the statement of Stokes' theorem.

Vector surface integrals. Vector surface integrals are defined similarly to scalar surface integrals. Whereas scalar surface integrals are defined in terms of the scalar differential dS, which is $\|\mathbf{N}\|\,ds\,dt$ in terms of dS, vector surface integrals are defined in terms of the vector differential $d\mathbf{S} = \mathbf{N}\,ds\,dt$.

Precisely, if $\mathbf{F}$ is a vector field in $\mathbf{R}^3$, then the vector surface integral is defined as

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\,ds\,dt.$$

The normal vector $\mathbf{N}$ doesn't have to be a unit normal vector. We can make it a unit vector by dividing it by its length. Let $\mathbf{n}$ denote the *unit normal vector*.

$$\mathbf{n}(s,t) = \frac{\mathbf{N}(s,t)}{\|\mathbf{N}(s,t)\|}$$

Then $\mathbf{N} = \|\mathbf{N}\|\,\mathbf{n}$, so we can rewrite the vector surface integral as

$$\begin{aligned} \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}) \cdot \mathbf{N}\,ds\,dt \\ &= \iint_D (\mathbf{F}(\mathbf{X}) \cdot \mathbf{n})\,\|\mathbf{N}\|\,ds\,dt \\ &= \iint_D (\mathbf{F} \cdot \mathbf{n})\,dS \end{aligned}$$

In other words, the vector differential $d\mathbf{S}$ is the product of the unit normal vector $\mathbf{n}$ and the scalar differential dS.

Example 1. Exercise 2 in section 7.2 has both a scalar surface integral and a vector surface integral. The surface is the same for both.

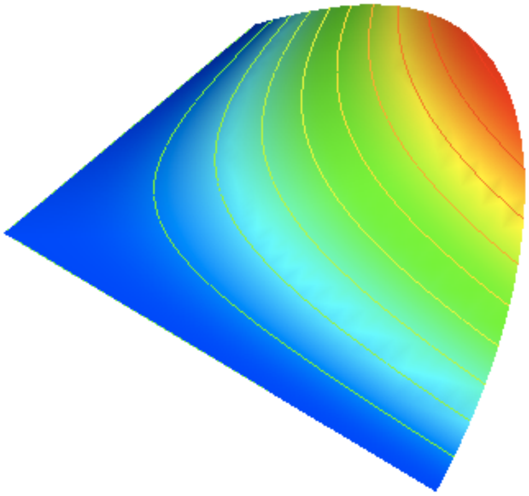
Let D be the quarter disk

$$D = \{(s,t) | s^2 + t^2 \leq 1, s \geq 0, t \geq 0\},$$

and let the surface parametrization $\mathbf{X} : D \to \mathbf{R}^3$ be defined by

$$\mathbf{X}(s,t) = (s+t, s-t, st).$$

The surface looks like it's a quarter disk in the xy-plane, but lifted up somewhat in the z-direction. One edge is on the line $y = x$ in the xy-plane, another on the line $y = -x$.



For either a scalar surface integral or a vector surface integral, we'll need to calculate the normal vector $\mathbf{N}$. Let's use the expression for $\mathbf{N}$ in terms of Jacobians this time.

$$\mathbf{N} = \frac{\partial(y,z)}{\partial(s,t)}\mathbf{i} - \frac{\partial(x,z)}{\partial(s,t)}\mathbf{j} + \frac{\partial(x,y)}{\partial(s,t)}\mathbf{k}$$

$$\begin{aligned}
\frac{\partial(y,z)}{\partial(s,t)} &= \frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s} = 1s - (-1)t = s+t \\
\frac{\partial(x,z)}{\partial(s,t)} &= \frac{\partial x}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial z}{\partial s} = 1s - 1t = s-t \\
\frac{\partial(x,y)}{\partial(s,t)} &= \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} = 1(-1) - 1(1) = -2
\end{aligned}$$

Therefore, $\mathbf{N} = (s+t, t-s, -2)$. The length of $\mathbf{N}$ is

$$\|\mathbf{N}\| = \sqrt{(s+t)^2 + (t-s)^2 + 4} = \sqrt{2s^2 + 2t^2 + 4}.$$

First, let's evaluate the scalar surface integral $\iint_{\mathbf{X}} f\,dS$, where $f(x,y,z) = 4$. Polar coordinates help here, and later the substitution, $u = r^2 + 2$, $du = 2r\,dr$, helps.

$$\begin{aligned}
&\iint_{\mathbf{X}} 4\,dS \\
&= \iint_D 4\,\|\mathbf{N}(s,t)\|\,ds\,dt \\
&= \iint_D 4\sqrt{2s^2+2t^2+4}\,ds\,dt \\
&= 4\sqrt{2}\iint_D \sqrt{s^2+t^2+2}\,ds\,dt \\
&= 4\sqrt{2}\int_0^{\pi/2}\int_0^1 \sqrt{r^2+2}\,r\,dr\,d\theta \\
&= 2\sqrt{2}\int_0^{\pi/2}\int_2^3 \sqrt{u}\,du\,d\theta \\
&= 2\sqrt{2}\int_0^{\pi/2} \tfrac{2}{3}(3^{3/2} - 2^{3/2})\,d\theta \\
&= \tfrac{2}{3}\pi\sqrt{2}\,(3^{3/2} - 2^{3/2})
\end{aligned}$$

Next, let's evaluate the vector surface integral $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

The definition for vector surface integrals says

$$\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X})\cdot\mathbf{N}\,ds\,dt.$$

We already have $\mathbf{N}$. So

$$\begin{aligned}
\mathbf{F}(\mathbf{X})\cdot\mathbf{N} &= (x,y,z)\cdot(s+t, t-s, -2) \\
&= (s+t, s-t, st)\cdot(s+t, t-s, -2) \\
&= s^2 + 2st + t^2 - s^2 + 2st - t^2 - 2st \\
&= 2st.
\end{aligned}$$

Therefore, the vector surface integral equals

$$\iint_D 2st\,ds\,dt,$$

which, in polar coordinates, equals

$$\begin{aligned}
&\int_0^{\pi/2}\int_0^1 2r^2\sin\theta\cos\theta\,r\,dr\,d\theta \\
&= \left(\int_0^1 r^2 r\,dr\right)\left(\int_0^{\pi/2} 2\sin\theta\cos\theta\,d\theta\right) \\
&= \left(\tfrac{1}{4}r^4\Big|_0^1\right)\left(\sin^\theta\Big|_0^{\pi/2}\right) = \tfrac{1}{4}
\end{aligned}$$

Vector surface integrals as flux. One nice physical interpretation of the vector surface integral $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$ is as a measure of the flow of a fluid across the surface S. Suppose a fluid is moving in space, and $\mathbf{F}(x, y, z)$ is the velocity of that fluid at the point (x, y, z). The dot product $\mathbf{F} \cdot \mathbf{n}$, where $\mathbf{n}$ is the outward unit normal vector to the surface S, gives the component of the velocity of the fluid going out through the surface S. So, if $\mathbf{F}$ is perpendicular to $\mathbf{n}$, then there is no fluid going through the surface, but if $\mathbf{F}$ is in the same direction as $\mathbf{n}$, then the fluid is going through the surface at velocity $\mathbf{F}$. The vector surface integral sums all the velocities (some can be negative indicating inward flow, some positive indicating outward flow) to give the total velocity through the entire surface S. This is called the *flux* of $\mathbf{F}$ across S.

The statement of Stokes' theorem. Let a surface S in space have a boundary ∂S. This boundary may have one component, but it may have more than one. All the boundaries have to be oriented in the same way. Let $\mathbf{F}$ be a vector field in $\mathbf{R}^3$. Stokes' theorem equates a surface integral to a line integral. It says that the curl of $\mathbf{F}$ integrated over the surface S equals the vector field $\mathbf{F}$ integrated over the boundary ∂S.

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

In the special case when the surface S lies in the (x, y)-plane, we've already seen this is a consequence of Green's theorem. The general statement of Stokes' theorem allows the surface S to be in space.

Example 2 (Stokes' theorem)**.** Let S be the surface parametrized by

$$\mathbf{X}(s, t) = (s \cos t, s \sin t, t)$$

for $0 \leq s \leq 1$ and $0 \leq t \leq \pi/2$, and let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y, z) = (z, x, y).$$

We'll verify Stokes' theorem.

You probably recognize the surface S as a $\frac{1}{4}$-turn of a helicoid where the angle t varies from 0 to 90 d

Let's evaluate the surface integral $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ first. We'll need the normal vector $\mathbf{N}$.

$$\begin{aligned} \mathbf{N}(s,t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s \sin t & s \cos t & 1 \end{vmatrix} \\ &= (\sin t, -\cos t, s) \end{aligned}$$

We'll also need the curl of $\mathbf{F}$.

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} \\ &= (1, 1, 1) \end{aligned}$$

Then

$$
\begin{aligned}
&\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}\\
=\;&\iint_D \nabla\times\mathbf{F}\cdot\mathbf{N}\,ds\,dt\\
=\;&\iint_D (1,1,1)\cdot(\sin t,-\cos t,s)\,ds\,dt\\
=\;&\int_0^{\pi/2}\int_0^1(\sin t-\cos t+s)\,ds\,dt\\
=\;&\int_0^{\pi/2}(\sin t-\cos t+\tfrac12)\,dt=\frac{\pi}{4}
\end{aligned}
$$

Now let's evaluate the line integral $\oint_{\partial S}\mathbf{F}\cdot d\mathbf{s}$. The boundary ∂S comes in four parts. First, the line segment from the origin to $(1,0,0)$ which we can parametrize by $\mathbf{x}_1(t)=(t,0,0)$ for $0\le t\le 1$. Second, $\mathbf{x}_2$: when $s=1$, the helix $(\cos t,\sin t,t)$ for $0\le t\le\pi/2$. Third, the line segment from $(0,1,pi/2)$ back to the z-axis at $(0,0,\pi/2)$, which we can parametrize by $\mathbf{x}_3(t)=(0,1-t,\pi/2)$. And fourth, the line segment from $(0,0,\pi/2$ back to the origin, which we can parametrize by $\mathbf{x}_4(t)=(0,0,\pi/2-t)$ for $0\le t\le\pi/2$.

We'll evaluate the integrand over each of the four parts of the boundary, then add the resulting values to get the integral over all of ∂S.

$$
\begin{aligned}
\int_{\mathbf{x}_1}\mathbf{F}\cdot d\mathbf{s} &= \int_{\mathbf{x}_1}\mathbf{F}(\mathbf{x}_1(t))\cdot\mathbf{x}_1'(t)\,dt\\
&= \int_0^1 (0,t,0)\cdot(1,0,0)\,dt\\
&= \int_0^1 0\,dt=0
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbf{x}_2}\mathbf{F}\cdot d\mathbf{s} &= \int_{\mathbf{x}_2}\mathbf{F}(\mathbf{x}_1(t))\cdot\mathbf{x}_2'(t)\,dt\\
&= \int_0^{\pi/2}(t,\cos t,\sin t)\cdot(-\sin t,\cos t,1)\,dt\\
&= \int_0^{\pi/2}(-t\sin t+\cos^2 t+\sin t)\,dt\\
&= \pi/4
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbf{x}_3}\mathbf{F}\cdot d\mathbf{s} &= \int_{\mathbf{x}_3}\mathbf{F}(\mathbf{x}_3(t))\cdot\mathbf{x}_3'(t)\,dt\\
&= \int_0^1(\pi/2,0,1-t)\cdot(0,-1,0)\,dt\\
&= \int_0^1 0\,dt=0
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbf{x}_4}\mathbf{F}\cdot d\mathbf{s} &= \int_{\mathbf{x}_4}\mathbf{F}(\mathbf{x}_4(t))\cdot\mathbf{x}_4'(t)\,dt\\
&= \int_0^1(\pi/2-t,0,0)\cdot(0,0,-1)\,dt\\
&= \int_0^1 0\,dt=0
\end{aligned}
$$

Thus, $\iint_S\nabla\times\mathbf{F}\cdot d\mathbf{S}=\pi/4$, and Stokes' theorem is verified.

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