Speed, velocity, and acceleration Math 131 Multivariate Calculus

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We'll discuss on paths, that is, moving points. There isn't much to the concept of path. We're primarily interested in the first and second derivatives of paths, called velocity and acceleration, respectively.

The major illustration of these concepts in the text is a derivation of Kepler's laws of planetary motion from Newton's laws of gravity. We'll look at that next.

Paths and parameterized curves. A function $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ can be interpreted in different ways. You can think of it as giving the position $\mathbf{x}(t)$, also called the position vector, of a point at time t. Or you can think of it as a way of parameterizing a curve where for each value of t a different point $\mathbf{x}(t)$ on the curve is named. Either way, we'll call $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ a *path* in $\mathbf{R}^n$.

Frequently, the domain isn't all of $\mathbf{R}$ but just a finite interval, $I = [a, b]$ with endpoints a and b. In that case $\mathbf{x}(a)$ is the initial position, and $\mathbf{x}(b)$ is the final position.

We generally treat the path $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ as a dynamic thing, but its image, which is a curve, a particular subset of $\mathbf{R}^n$, as static thing.

For the discussion that follows, $\mathbf{x}$ denotes one of these functions $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ which we'll variously refer to as a path, moving point, position function, or position vector.

Velocity, speed, and acceleration. We call the derivative of $\mathbf{x}$ its *velocity*, and often denote it $\mathbf{x}'$. This is pretty much Newton's notation although he used a dot above the variable instead of a prime. Its second derivative $\mathbf{x}''$ is its *acceleration*. Both velocity and acceleration are vectors. We define its *speed*, $\|\mathbf{x}'\|$, to be the length of velocity, so speed is a nonnegative scalar.

So, at a time t, an object has a position $\mathbf{x} = \mathbf{x}(t)$; a velocity $\mathbf{x}' = \mathbf{x}'(t)$, which is sometimes denoted $\mathbf{v} = \mathbf{v}(t)$; an acceleration $\mathbf{x}'' = \mathbf{x}''(t)$, which is sometimes denoted $\mathbf{a} = \mathbf{a}(t)$; and a speed $|\mathbf{x}'|$.

Example 1. Consider the moving point in the plane

$$\mathbf{x} = (x, y) = (t^4 - 2t^2, t^3 - 2t)$$

defined for t in the interval $[-1.5, 1.5]$. It's displayed in figure 1. This $\mathbf{x}$ is a moving point, that is, a path. The initial position when $t = -1.5$ is the point $(0.56, 0.37)$, and the final position when $t = 1.5$ is the point $(0.56, -0.37)$. The point passes through the origin three times, first when $t = -\sqrt{2}$, then when $t = 0$, and again when $t = \sqrt{2}$.

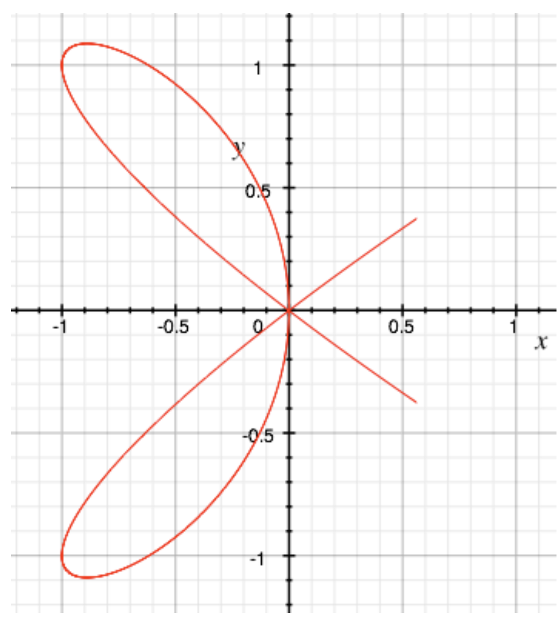

Figure 1: Path $\mathbf{x} = (t^4 - 2t^2, t^3 - 2t)$

Its image is the curve

$$\{(t^4 - 2t^2, t^3 - 2t) \mid -1.5 \le t \le 1.5\}.$$

If you solve the pair of equations $x = t^4 - 2t^2$ and $y = t^3 - 2t$ simultaneously for x and y and eliminate t in the process, then you'll get the single equation $y^4 = x(x^2 - 2y^2)$. This image is, therefore, part of this curve.

The velocity at time t is

$$\mathbf{v} = \mathbf{x}' = (4t^3 - 2t, 3t^2 - 2)$$

and the acceleration is

$$\mathbf{a} = \mathbf{x}'' = (12t^2 - 2, 6t).$$

The speed at time t is

$$\begin{aligned} \|\mathbf{x}'\| &= \|(4t^3 - 2t, 3t^2 - 2)\| \\ &= \sqrt{(4t^3 - 2t)^2 + (3t^2 - 2)^2} \end{aligned}$$

Example 2. Consider the moving point in space

$$\mathbf{x} = (x, y, z) = (\cos 3t, \sin 4t, \sin 5t)$$

defined for t in the interval $[0, 2\pi]$. It's displayed in figures 2 and 3 from slightly different perspectives with the curves thickened and colored so you can see them better. The initial position when $t = 0$ and the final position is the same, namely the point $(1, 0, 0)$. The point swings around and around the origin like a crazy roller coaster.

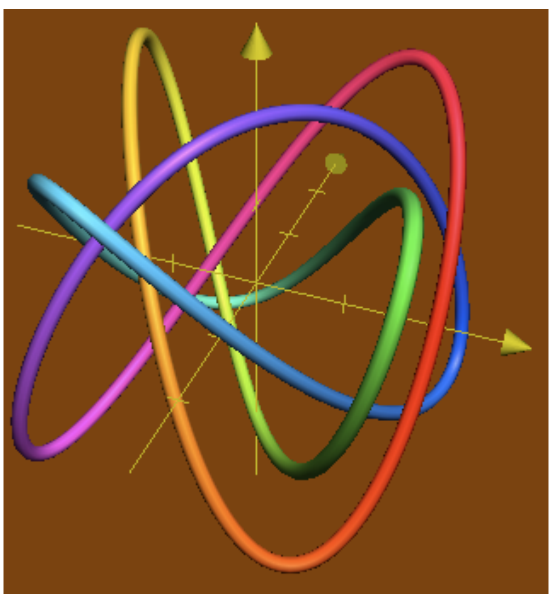

Figure 2: $\mathbf{x} = (\cos 3t, \sin 4t, \sin 5t)$

It's velocity at time t is

$$\mathbf{v} = \mathbf{x}' = (-3\sin 3t, 4\cos 4t, 5\cos 5t)$$

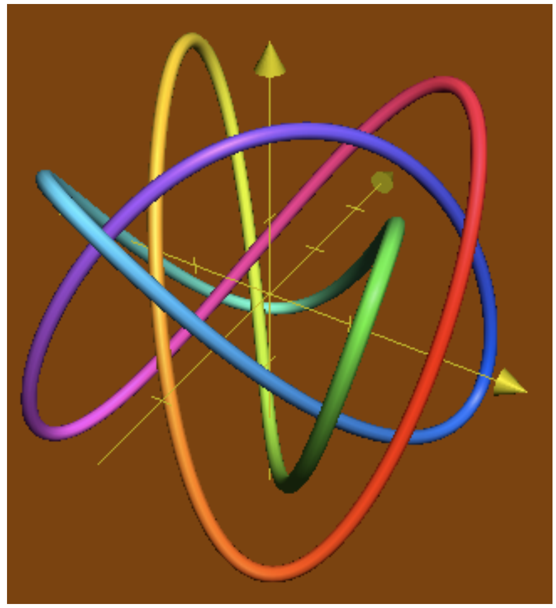

Figure 3: $\mathbf{x} = (\cos 3t, \sin 4t, \sin 5t)$

and the acceleration is

$$\mathbf{a} = \mathbf{x}'' = (-9\cos 3t, -16\sin 4t, -25\sin 5t).$$

The speed at time t is

$$\begin{aligned} \|\mathbf{x}'\| &= \|(-3\sin 3t, 4\cos 4t, 5\cos 5t)\| \\ &= \sqrt{9\sin^2 3t + 16\cos^2 4t + 25\cos^2 5t} \end{aligned}$$

Velocities and tangent lines. The tangent line at a point $\mathbf{x}_0 = \mathbf{x}(t_0)$ on a path $\mathbf{x}$ is the straight line passing through $\mathbf{x}_0$ in the direction of the velocity $\mathbf{v}_0 = \mathbf{x}'(t_0)$. A parametric equation for this line is

$$\mathbf{L}(t) = \mathbf{x}_0 + t\mathbf{v}_0$$

where t is a scalar variable.

When drawing velocity vectors for a moving point, they're usually drawn at the point, and that means they will fall along the tangent lines in the direction the point is moving.

Tangents on a circular path. Let's take, for example, a path $\mathbf{u} : \mathbf{R} \to \mathbf{R}^2$ on the unit circle, that is, $\|\mathbf{u}(t)\| = 1$ for all t. We'll show that the velocity vector $\mathbf{u}'$ is orthogonal to the position vector $\mathbf{u}$, as you should expect since the velocity vector ought to point in a direction tangent to the circle. Since $\mathbf{u} = (x, y)$ is a unit vector, therefore $x^2 + y^2 = 1$. Differentiate that equation with respect to t to get $2xx' + 2yy' = 0$. Therefore, the dot product, $\mathbf{u} \cdot \mathbf{u}'$, which is $(x, y) \cdot (x', y')$, is 0. Thus, $\mathbf{u}$ and $\mathbf{u}'$ are orthogonal. Note that this is true whether the object is moving uniformly or nonuniformly around the circle.

Vector product rules. Although there is no multiplication of vectors exactly corresponding to multiplication of scalars, there are two vector products, namely, dot product and cross product. These have rules of differentiation like the product rule when the arguments $\mathbf{x}$ and $\mathbf{y}$ are functions of one variable.

$$\boxed{\begin{array}{rcl} (\mathbf{x} \cdot \mathbf{y})' & = & \mathbf{x}' \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y}' \\ (\mathbf{x} \times \mathbf{y})' & = & \mathbf{x}' \times \mathbf{y} + \mathbf{x} \times \mathbf{y}' \end{array}}$$

Note that the order of the factors is important for the cross product since it's not commutative.

These rules follow directly from the usual product rule and the definitions of dot and cross product. For example, here's the proof of the dot product rule in dimension 2.

$$\begin{array}{rcl} (\mathbf{x} \cdot \mathbf{y})' & = & \big((x_1, x_2) \cdot (y_1, y_2)\big)' \\ & = & (x_1 y_1 + x_2 y_2)' \\ & = & x_1' y_1 + x_1 y_1' + x_2' y_2 + x_2 y_2' \\ & = & \mathbf{x}' \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{y}' \end{array}$$

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