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Independent events
Math 217 Probability and Statistics
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Independent events. Independence is an intuitive idea that we can easily formalize. The phrase "the events E and F are independent" means that the knowledge that E has occurred has no effect on whether or not F will occur, and vice versa. Now that we have the concept of conditional probability, we can take that to mean that $P(F|E) = P(F)$ and $P(E|F) = P(E)$. A little algebra shows that each of those conditions is equivalent to $P(E \cap F) = P(E)\,P(F)$. So we'll take that as our definition of independent events.

Definition. Events E and F are said to be *independent* if

$$P(E \cap F) = P(E)\,P(F),$$

otherwise they're said to be *dependent*.

Note that E and F are independent if and only if E and F^c are independent. That is, if the event E is independent of F, then it's also independent of its complement F^c. Likewise E^c is independent of both F and F^c. Thus, if either E or E^c is independent of either F or F^c, then both E and E^c are independent of both F and F^c.

You can also state independence in terms of conditional probabilities. Events E and F are independent when $P(E) = P(E|F)$ or when $P(F) = P(F|E)$. Intuitively, the condition that the event F has occurred doesn't affect the probability that the event E will occur.

Product spaces. An important situation for independent events arises from taking the product two sample spaces Ω_1 and Ω_2. We create a new sample space, called the *product* or *Cartesian product*, $\Omega_1 \times \Omega_2$, whose outcomes are ordered pairs (x_1, x_2) of outcomes, $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$. Probabilities are assigned to the events in this product sample space so that, for $A_1 \subseteq \Omega_1$ and $A_2 \subseteq \Omega_2$, the probability of a product event

$$A_1 \times A_2 = \{(s_1, s_2) \mid s_1 \in A_1 \text{ and } x_2 \in A_2\}$$

is $P(A_1 \times A_2) = P(A_1)\,P(A_2)$.

Frequently, Ω_1 and Ω_2 are the same sample space Ω, and in that case the product sample space $\Omega \times \Omega$ is denoted Ω^2.

Of course, products of more than two sample spaces are defined similarly.

We've seen some examples of product spaces before we had this definition. For example, if Ω is the 6-outcome sample space for a fair die, then Ω^2 is the 36-element sample space for a pair of dice. For another example, take the uniform continuous distribution on the unit interval $\Omega = [0, 1]$. Then the uniform continuous distribution on the unit square $\Omega^2 = [0, 1] \times [0, 1]$ is the product space.

We'll frequently have use of the n^{th} power of a sample space, Ω^n.

Independence of more than two events. Sometimes you'll want to express the situation where none of the three events E, F, and G affect each other, and, furthermore, combinations of two of them don't affect the third. Here's the proper definition to reflect that situation.

Definition. Events E, F, and G are said to be *jointly independent* if each pair of them are independent, and $P(E \cap F \cap G) = P(E)\,P(F)\,P(G)$.

In that case, their complements in various combinations will also be jointly independent.

Of course there is analogous definition for the independence of more than three events, even a countably infinite number of events. $E_1, E_2, \ldots$ are independent when the probability of an intersection of any number of them is the product of their probabilities.

Joint random variables. When we use the notation of joint random variables, we're implicitly using product spaces. For example, if X is the outcome for one fair die, and Y the outcome for another fair die, then (X, Y) is the joint random variable for the outcome of the pair of dice. It's a random variable on the product space S^2 having 36 elements described above.

If you have two such discrete random variables X and Y, they're said to be *independent* if for all their outcomes x and y, $P(X{=}x \text{ and } Y{=}y) = P(X{=}x)\,(Y{=}y)$. We'll discuss independence of random variables later in detail.

Random samples, that is, independently and identically distributed random variables (i.i.d). It's too early to define this concept precisely, but we've already looked at example of random samples. They're a central concept in probability and statistics, so we'll begin using the terminology right away.

If you flip many fair coins, or the same fair coin repeatedly, then the sample space for the experiment is the product space Ω^n where Ω is the sample space for one coin flip, $\Omega = \{H, T\}$. It's appropriate to take Ω^n to be the sample space because we assume that the outcome of one coin flip, called a *trial*, is not affected by the outcome of any of the other coin flips.

(This may not be accurate since when you repeatedly flip one coin, the outcome of the current flip actually does affect the outcome of the next one. Experiments have shown that about 51% of the time, the succeeding coin flip will have the same outcome.)

When we have n independent trials X_1, X_2, ..., X_n of the same experiment, the joint random variable $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is a random variable on a product space. We'll call this n-tuple $\mathbf{X}$ a *random sample* and we'll say X_1, X_2, ..., X_n are *identically distributed and independent* random variables, abbreviated i.i.d.

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