Math 225 Modern Algebra
Final Exam

Prof. D. Joyce

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On this take-home exam you may consult your notes for the course, the text book, and any other books you like. Do all your own work; don't consult with anyone but me. Start your answer to each problem on a separate page. Staple the pages together before you hand them in. Points are in square brackets.

Problem 1. On composition and conjugation in D_4. [20 points; 5 each part]

Consider the group D_4 of symmetries of the square. Let ρ be the rotation 90 degrees counterclockwise, and let ϕ be a reflection across a vertical axis.

a. Describe the composition $\rho\phi$. (Is it a rotation or reflection? If rotation, then by how much? If reflection, what's the axis of reflection?)

b. Describe the composition $\phi\rho$.

c. Describe the conjugate $\rho^{-1}\phi\rho$ of ϕ.

d. Describe the conjugate $\phi^{-1}\rho\phi$ of ρ.

Problem 2. On basic properties of groups. [20; 10 points each part]

Recall the definition of group that we've used in this course. A *group* is a set G together with a binary operation that satisfies the following three axioms:

(1) Associatvity. $\forall x, y, z$, $(xy)z = x(yz)$.

(2) Identity. $\exists x$, $\forall y$, $xy = y = yx$. Such an x is called an identity element.

(3) Inverses. $\forall x$, $\exists y$, $xy = yx = 1$, where 1 is an identity element as described in axiom 2.

a. Note that axiom 2 says that there is *at least* one identity element. Prove that the identity element is unique. Use only the three axioms for groups mentioned in the definition, and point out every time you use an axiom in your proof.

b. Prove the cancellation law for groups: if x, y, and z are elements in a group G, then $xz = yz$ implies $x = y$. Use only the three axioms for groups mentioned in the definition and the results of part a (which allow you to denote the unique group identity as 1) in your proof. Point out every time you use an axiom or part a in your proof.

Problem 3. On rings, ideals, and quotients [20; 10 points each part]

Let R be a commutative ring. Let N be the subset of R consisting of all elements x for which some power is 0. That is

$$N = \{x \mid \exists n, x^n = 0\}.$$

a. Prove that N is an ideal of R.

b. Consider the quotient ring R/N. Prove that if an element $[x] \in R/N$ has some power equal to $[0]$, that is, if $[x]^m = [0]$ for some m, then $[x] = [0]$. (In an alternate notation, $(x + N)^m = N$ implies $x + N = N$. Use whichever notation you prefer.)

Problem 4. On irreducible polynomials. [20]

Consider the polynomial $f(x) = x^6 + 3x^2 - 9x + 3$ in $\mathbf{Q}[x]$. Prove that $f(x)$ is an irreducible polynomial.

Problem 5. On orders of groups and elements in groups. [20; 5 points each part]

a. What is the order of the symmetric group S_6?

b. Consider the group D_{10}, the symmetry group of a regular decagon (10-sided polygon). One of the elements in this group is a rotation by 108 degrees. What is the order of that element?

c. Consider the permutation $(135496)(2870)$. What is its order?

d. What is the order of the alternating group A_6?