1. [20] On fields. Recall the definition of a field. A field $F$ consists of

1. a set, also denoted $F$ and called the underlying set of the field;

2. a binary operation $+: F \times F \to F$ called addition, which maps an ordered pair $(x, y) \in F \times F$ to its sum denoted $x + y$;

3. another binary operation $\cdot : F \times F \to F$ called multiplication, which maps an ordered pair $(x, y) \in F \times F$ to its product denoted $x \cdot y$, or more simply just $xy$; such that

4. addition is commutative, that is, for all elements $x$ and $y$, $x + y = y + x$;

5. multiplication is commutative, that is, for all elements $x$ and $y$, $xy = yx$;

6. addition is associative, that is, for all elements $x, y$, and $z$, $(x + y) + z = x + (y + z)$;

7. multiplication is associative, that is, for all elements $x, y$, and $z$, $(xy)z = x(yz)$;

8. there is an additive identity, an element of $F$ denoted 0, such that for all elements $x$, $0 + x = x$;

9. there is a multiplicative identity, an element of $F$ denoted 1, such that for all elements $x$, $1x = x$;

10. there are additive inverses, that is, for each element $x$, there exists an element $y$ such that $x + y = 0$; such a $y$ is called the negation of $x$;

11. there are multiplicative inverses of nonzero elements, that is, for each nonzero element $x$, there exists an element $y$ such that $xy = 1$; such a $y$ is called a reciprocal of $x$;

12. multiplication distributes over addition, that is, for all elements $x, y$, and $z$, $x(y + z) = xy + xz$; and

13. $0 \neq 1$.

Carefully prove that 0 times any element in a field is 0, $0x = 0$, using only the definition above and no other properties of a field (unless you prove them as well). Justify every statement and equation. Write full sentences.
2. [15; 5 points each part] On rings.
   a. Give an example of a ring \( R \) and two elements \( x \) and \( y \) in \( R \), neither of which is 0, but the product \( xy \) of the two elements is 0.
   b. Give an example of a ring of characteristic 0.
   c. Give an example of a subring of the field \( \mathbb{R} \) of real numbers other than \( \mathbb{R} \) itself.

3. [20; 5 points each part] On groups. For each of the following, state if it is a group or not. If not, explain why not, but if so, you don’t have to give a reason why.
   a. The set \( \{1, -1, i, -i\} \) of four complex numbers under addition.
   b. The set \( \{1, -1, i, -i\} \) of four complex numbers under multiplication.
   c. The set of six functions including \( f(x) = x \), \( g(x) = 1 - x \), \( h(x) = \frac{1}{1-x} \), \( i(x) = x \), \( k(x) = \frac{x-1}{x} \), and \( \ell(x) = \frac{x}{x-1} \) under composition.
   d. The set of \( 2 \times 2 \) matrices in \( M_2(\mathbb{R}) \) with positive determinants under matrix multiplication.

4. [16; 8 points each part] On number theory.
   a. Draw a Hasse diagram of the divisors of 30.
   b. Use the Euclidean algorithm to show that the greatest common divisor of 105 and 154 is 7. Show your work.

5. [15] On ordered fields. Recall that an order on a field \( F \) is determined by a subset \( P \) whose elements are called positive such that (1) \( F \) is partitioned into three parts: \( P \), \( \{0\} \), and \( N = \{x \in F \mid x \in P\} \), (2) the sum of two positive elements is positive; and (3) the product of two positive elements is positive.
   Explain in your own words why a field of prime characteristic \( p \) cannot have an order of this kind.

6. [16; 8 points each part] On finite fields. We have had examples and exercises on finite fields. The Galois field \( GF(2) \) is the ring \( \mathbb{Z}_2 \) of integers modulo 2. In this exercise you’ll construct the Galois field \( GF(8) \) as an extension of \( \mathbb{Z}_2 \).
   a. Find at least one of the following cubic polynomials that has no root in \( \mathbb{Z}_2 \): \( x^3, x^3 + 1, x^3 + x, x^3 + x + 1, x^3 + x^2, x^3 + x^2 + 1, x^3 + x^2 + x, x^3 + x^2 + x + 1 \). That is to say, if \( f(x) \) is the polynomial, its value at neither of the two elements of \( \mathbb{Z}_2 \) is equal to 0.
   
   Now let \( f(x) \) be that polynomial you found in part a. Let \( F \) be the 3-dimensional vector space over \( \mathbb{Z}_2 \) of 8 elements where each element is written as \( ax^2 + bx + c \) with \( a, b, \) and \( c \) each in \( \mathbb{Z}_2 \). Define multiplication on \( F \) so that \( f(x) = 0 \). (So, for instance, if \( f(x) = x^3 + x^2 + x + 1 \), then \( x^3 = -x^2 - x - 1 \).)
   b. With your choice of \( f(x) \), \( F \) will be a field where every nonzero element has a reciprocal. Determine the reciprocal of \( x \) in \( F \), that is, find some polynomial whose product with \( x \) is equal to 1 modulo \( f(x) \).