CLARK UNIVERSITY

Exercises
Math 225 Modern Algebra
Fall 2017

Exercise 1. On properties of operations.

(a). Is the binary operation $x * y = \dfrac{x+y}{xy}$ for positive x and y a commutative operation? That is, is it true that $x * y = y * x$ for all positive x and y? Is it associative? Explain your answer.

For commutativity, we need to check whether $x*y = y*x$ for positive x and y. That means we need to check the equation

$$\frac{xy}{x+y} = \frac{yx}{y+x}.$$

That works because addition and multiplication of real numbers is commutative.

For associativity, we need to check whether $(x*y)*z = x*(y*z)$ for positive x and y. That means we need to check the equation

$$\frac{(x*y)z}{(x*y)+z} = \frac{x(y*z)}{x+(y*z)}$$

which expands to the equation

$$\frac{\dfrac{xy}{x+y}\,z}{\dfrac{xy}{x+y}+z} = \frac{x\,\dfrac{yz}{y+z}}{x+\dfrac{yz}{y+z}}.$$

When you simplify the left side by multiplying both the numerator and the denominator by $x+y$, you'll get $\dfrac{xyz}{xy+xz+yz}$. You'll get the same thing when you multiply the numerator and denominator of the right side by $y+z$. So associativity checks out.

(b). Is it true that $(w-x)-(y-z) = (w-y)-(x-z)$ is an identity for real numbers? Can you say why or why not? (The word "identity" is used for an equation which holds whenever both sides of the equation are defined and are equal.)

Yes. The left side is $w-x-y+z$ while the right side is $w-y-x+z$, which, of course are equal. This is one of the charactering properties of subtraction.

(c). Although multiplication in $\mathbf{R}$ distributes over addition, addition doesn't distribute over multiplication. Give an example where it doesn't.

For addition to distribute over multiplication, the equation $x+(yz) = (x+y)(x+z)$ would have to be true for all values of the variables. You can take almost any values to show that doesn't hold. For example, for $x=1$, $y=2$, and $z=3$, $x+(yz)=7$ while $(x+y)(x+z)=12$ which aren't equal. (Just don't take $x=0$, for then the equation is true.)

Note that in set theory, intersection distributes over union *and* union distributes over intersection. Both $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$ and $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$.

Exercise 2. On fields. None of the following are fields. In each case, the operations of addition and multiplication are the usual ones.

(a). The integers $\mathbf{Z}$ do not form a field. Why not?

It doesn't have multiplicative inverses. For example, the reciprocal of 3 is not an integer.

(b). The positive real numbers $\{x\in\mathbf{R}\,|\,x>0\}$ do not form a field. Why not?

It doesn't have an additive identity; also no additive inverses (negations).

(c). The set of real numbers between -10 and 10, that is,

$$(-10,10) = \{x\in\mathbf{R}\,|\,-10<x<10\}$$

is not a field. Why not?

It's not closed under addition or multiplication. For example if you add or multiply 7 and 8, you don't get a number in that interval.

Math 225 Home Page at
`http://aleph0.clarku.edu/~djoyce/ma225/`