CLARK UNIVERSITY

Exercises
Math 225 Modern Algebra
Fall 2017

Formal proofs are not required for these exercises, but convincing arguments should be supplied.

7. Prove that if $f : A \to B$ is a function between two finite sets of the same cardinality, then the following three conditions are equivalent: (1) f is a bijection, (2) f is an injection, and (3) f is a surjection.

Note that (1) implies both (2) and three since a bijection is defined as being both an injection and a surjection. What remains to be shown is that a surjection between two sets of the same finite cardinality is also an injection, therefore a bijection, and that injection between two sets of the same finite cardinality is also an surjection, therefore a bijection.

Injective implies surjective: There are lots of explanations. Here's just one of many. Let $A = \{a_1, a_2, \ldots, a_n\}$. Let $b_i = f(a_i)$ for each i from 1 through n. Since f is an injection, therefore $b_1, b_2, \ldots, b_n$ are all distinct. Since all n elements of B are accounted for, therefore $B = \{b_1, b_2, \ldots, b_n\}$, and f is surjective. Q.E.D.

Surjective implies injective: Let $B = \{b_1, b_2, \ldots, b_n\}$. Since f is surjective, for each i from 1 through n, there is some element of A sent to b_i. Let one of those elements be denoted a_i. Since f is a function, therefore $a_1, a_2, \ldots, a_n$ are all distinct. Since all n elements of A are accounted for, therefore $A = \{a_1, a_2, \ldots, a_n\}$. Each of the a_i's is sent to a different b_i, therefore f is injective. Q.E.D.

8. Since the structure of rings is defined in terms of addition and multiplication, if f is a ring isomorphism, it will preserve structure defined in terms of them. Verify that f preserves 0, 1, negation, and subtraction.

Let $f : A \to B$ be a ring homomorphism.

f preserves 0: We're to show that $f(0) = 0$. Since f is a ring isomorphism and $0 + 0 = 0$, therefore $f(0) + f(0) = f(0)$. Subtracting $f(0)$ from each side of that equation, we conclude that $f(0) = 0$. Q.E.D.

f preserves 1: We're to show that $f(1) = 1$. This is more difficult since it needn't hold for homomorphisms, but it does hold for isomorphisms. Let 1 be the identity in B. Some element $x \in A$ is sent to 1, that is, $f(x) = 1$. Since $1x = x$ and f preserves multiplication, therefore $f(1)f(x) = f(x)$, but $f(x) = 1$, so $f(1) = 1$. Q.E.D.

(Note: you can easily show that $f(1)f(1) = f(1)$, but that's not enough to conclude that $f(1) = 1$ since in a ring, $aa = a$ need not imply $a = 1$.)

f preserves negation: We're to show that $f(-x) = -f(x)$. Since $x+(-x) = 0$ and f preserves addition, therefore $f(x)+f(-x) = 0$. Subtracting $f(x)$ from each side of the equation, it follows that $f(-x) = -f(x)$. Q.E.D.

f preserves subtraction: We're to show that $f(x - y) = f(x)-f(y)$. Since $(x-y)+y = x$, therefore $f(x-y)+f(y) = f(x)$, and so $f(x - y) = f(x) - f(y)$. Q.E.D.

9. Prove that if f is a ring isomorphism, then so is its inverse function $f^{-1} : B \to A$.

We know that f preserves addition and multiplication, and that f^{-1} is the inverse function of f. From those two properties we're to show that f^{-1} also preserves addition and multiplication.

Show $f^{-1}(x) + f^{-1}(y) = f^{-1}(x + y)$: Let $s = f^{-1}(x)$ and $t = f^{-1}(y)$. Then $f(s) = x$ and $f(t) = y$. So $f(s + t) = f(s) + f(t) = x + y$. Therefore, $s + t = f^{-1}(x + y)$, that is, $f^{-1}(x) + f^{-1}(y) = f^{-1}(x + y)$. Q.E.D.

Products are analogous; just change addition to multiplication in the preceding argument.

10. Prove that if $f : A \to B$ and $g : B \to C$ are both ring isomorphisms, then so is their composition $(g \circ f) : A \to C$.

A ring isomorphism is a bijection that preserves addition and multiplication. Since f and g are both bijections, so is their composition $g \circ f$.

Likewise, their composition preserves addition as shown by the equation

$$\begin{aligned}(g \circ f)(x + y) &= g(f(x + y)) \\ &= g(f(x) + f(y)) \\ &= g(f(x)) + g(f(y)) \\ &= (g \circ f)(x) + (g \circ f)(y)\end{aligned}$$

11. Prove that if a ring is isomorphic to a field, then that ring is a field.

A field has two properties that a ring lacks, namely, a field has commutative multiplication and a field has multiplicative inverses. So these are the two properties to show for the ring.

Let $f : R \to F$ be a ring isomorphism from the ring R to the field F. We're to show that R has commutative multiplication and has reciprocals of nonzero elements.

Commutative multiplication: Let x and y be elements of R. We're to show that $xy = yx$. We know that $f(x)f(y) = f(y)f(x)$ holds in the field F. And since the isomorphism f preserves multiplication, that means that $f(xy) = f(yx)$. Since f is a bijection, therefore $xy = yx$. Q.E.D.

Reciprocals of nonzero elements: Let $x \in R$ be nonzero. We're to show that there is some element y in R so that $xy = 1$. The element $f(x)$ cannot be 0 in F since $f(0) = 0$ and f is an injection. Therefore, its inverse, $\dfrac{1}{f(x)}$, exists in F. Since f is surjective, there is some element of R that is sent to $\dfrac{1}{f(x)}$; call it y. Then $f(y) = \dfrac{1}{f(x)}$. Now $f(xy) = f(x)f(y) = f(x)\dfrac{1}{f(x)} = 1$. Since f sends 1 to 1 (exercise 8) and sends xy to 1, and f is injective, therefore $xy = 1$. Thus, the nonzero element x of R has y as its reciprocal. Q.E.D.

12. Suppose that both A and B are written multiplicatively and that $f : A \to B$ is a group isomorphism. Prove that $f(1) = 1$ and $f(x^{-1}) = f(x)^{-1}$ for all $x \in A$.

The arguments for this exercise are the same as those for exercise 8 for 0 and negation except that the notation is multiplicative instead of additive.

13. Draw Hasse diagrams for the divisors of 30, 32, and 60.

The diagram for 30 looks like a cube, that for 32 is a vertical line, and that for 60 is can be found from that of 30 by extending one side.

Math 225 Home Page at
`http://aleph0.clarku.edu/~djoyce/ma225/`