Exercises
Math 225 Modern Algebra
Fall 2017

29. Prove the following four properties about positive and negative elements of an ordered field.

1. the sum of negative elements is negative

   Proof: Let x and y be negative elements. Then $-x$ and $-y$ are positive elements. Their sum, $(-x)+(-y)$, is positive. But $(-x)+(-y) = -(x+y)$. Therefore, $x+y$ is negative. Q.E.D.

2. the product of a negative element and a positive element is negative

   Proof: Let x be negative and y positive. Since x is negative, therefore $-x$ is positive. Then $(-x)y$ is also positive. But $(-x)y = -xy$, so xy is negative. Q.E.D.

3. the product of two negative elements is positive

   Proof: Let x and y be negative. Then $-x$ and $-y$ are positive, and so is their product $(-x)(-y)$. But $(-x)(-y) = xy$, so xy is positive. Q.E.D.

4. 1 is positive, and -1 is negative

   Proof: Since $1 \neq 0$ in a field, 1 has to be either positive or negative. But $1 \cdot 1 = 1$, so in either case 1 is positive. And since -1 is the negation of the positive element 1, therefore -1 is negative. Q.E.D.

30. Show that $\mathbf{C}$ is not an ordered field. Hint: show why i can't be positive, zero, or negative.

Since $i \cdot i = -1$ and -1 is negative, and we know the product of two positives is positive, the product of two negatives is positive, and $i \neq 0$, therefore i can't be positive, zero, or negative. Thus $\mathbf{C}$ is not an ordered field. Q.E.D.

31. Prove the following five properties about "less than".

1. Trichotomy: For each pair x, y, exactly one of the three relations $x < y$, $x = y$, or $x > y$ holds.

   Proof: Exactly one of the three conditions $y - x \in P$, $y - x = 0$, and $y - x \in N$ holds. In the first case $x < y$, in the second $x = y$, and in the third $x > y$. Q.E.D.

2. Transitivity: $x < y$ and $y < z$ imply $x < z$.

   Proof: If $x < y$ and $y < z$, then both $y - x$ and $z - y$ are positive, and so their sum $z - x$ is also positive. Therefore, $x < z$. Q.E.D.

3. If x is positive and $y < z$, then $xy < xz$.

   Proof: If x is positive and $y < z$, then $z - y$ is positive. Therefore, the product $x(z - y)$ is also positive. But that equals $xz - xy$, so $xy < xz$. Q.E.D.

4. If x is negative and $y < z$, then $xy > xz$.

   The proof is similar to the previous.

5. If $0 < x < y$, then $0 < \dfrac{1}{y} < \dfrac{1}{x}$.

   Unlike the previous parts, this has several steps. Let's analyze it to see how we can prove it.

   We want to show that $0 < \dfrac{1}{y}$, that is, $1/y$ is positive. We know $0 < y$, so y is positive. But $y \cdot \dfrac{1}{y}$ is equal to 1, a positive element, so $\dfrac{1}{y}$ has to be positive. (It can't be negative since a negative times a positive is negative.)

   We also want to show that $\dfrac{1}{y} < \dfrac{1}{x}$ which is equivalent to $\dfrac{1}{x} - \dfrac{1}{y}$ being positive. But that's equal to $\dfrac{y-x}{xy}$. We know that $\dfrac{1}{y}$ is positive, and so is $\dfrac{1}{x}$, so their product $\dfrac{1}{xy}$ is also positive. But $y - x$ is positive since $x < y$. Therefore the product $\dfrac{y-x}{xy}$ is also positive.

   If you want, you can turn this analysis into a synthetic proof where the logic is cleaner.

32. Show that $\mathbf{H}$ can be represented as a subring of the complex matrix ring $M_2(\mathbf{C})$ where

$$1 \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad i \leftrightarrow \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$
$$j \leftrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad k \leftrightarrow \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

so that a generic quaternion $a + bi + cj + dk$ corresponds to the matrix
$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}$$

The correspondence is injective, so all that has to be shown is that it's compatible with addition and multiplication.

Addition is straightforward. Consider the sum of two quaternions $(a+bi+cj+dk)+(a'+b'i+c'j+d'k)$. Does it correspond to the sum of the two matrices?

$$\begin{aligned} &\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} + \begin{bmatrix} a'+b'i & c'+d'i \\ -c'+d'i & a'-b'i \end{bmatrix} \\ = &\begin{bmatrix} a+a'+bi+b'i & c+c'+di+d'i \\ -c-c'+di+d'i & a+a'-bi-b'i \end{bmatrix} \end{aligned}$$

Yes, it does.

Multiplication is a bit messier. Consider the product of two quaternions $(a+bi+cj+dk)(a'+b'i+c'j+d'k)$. Does it correspond to the sum of the two matrices? A bit of work will show that it does.

Math 225 Home Page at
`http://aleph0.clarku.edu/~djoyce/ma225/`