Trigonometric limits
Math 120 Calculus I
Fall 2015

Trigonometry is used throughout mathematics, especially here in calculus. The key to trig in calc is finding the derivatives of the sine and cosine functions. Almost everything else follows from those. Derivatives are defined in terms of limits, so that means we need to know something about limits and trig functions

The derivative of the sine function. Let's see what limits we need. The derivative of a function f is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

When $f(x) = \sin x$, that gives

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}.$$

We can't take this limit yet because of the h in the denominator. We can rewrite the numerator using the sum formula for sines

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

and we find

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin x\cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x\sin h}{h} \\ &= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\ &= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \end{aligned}$$

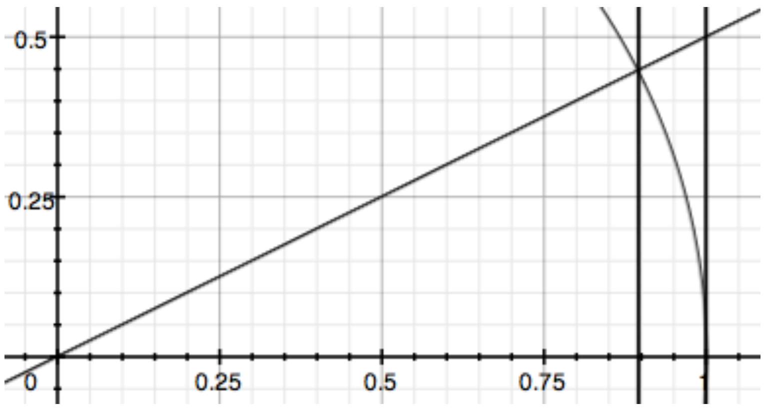
Note that we used a couple of properties of limits to rewrite the expression, in particular, the limit of a sum is the sum of the limits, and the limit of a constant times a function is the constant times the limit of the function.

We're left with two limits to evaluate, $\displaystyle\lim_{h\to 0} \frac{\cos h - 1}{h}$, and $\displaystyle\lim_{h\to 0} \frac{\sin h}{h}$. We will evaluate those two limits, and we'll find that the first equals 0, while the second equals 1. It will then follow that

$$f'(x) = (\sin x)0 + (\cos x)1 = \cos x,$$

in other words, the derivative of $\sin x$ is $\cos x$. Using these same two limits along with the sum formula for cosines, you can show that the derivative of $\cos x$ is $-\sin x$.

The limit, $\displaystyle\lim_{h\to 0}\frac{\sin h}{h} = 1$**.** We'll use a geometric analysis involving areas of triangles and sectors of circles, and finish it off with the sandwich theorem. First, we'll only take positive h, so we're actually only looking at the right limit, but since $\dfrac{\sin(-h)}{-h} = \dfrac{\sin h}{h}$, that's enough.



Shown is an acute angle h at the origin along with a smaller right triangle, a sector of the unit circle that contains that triangle, and a larger right triangle that contains the sector.

The width of the small triangle is $\cos h$ and its height is $\sin h$, so its area is $\frac{1}{2}\sin h\cos h$.

The sector of the unit circle whose arc is h radians has area $h/2$.

The width of the large triangle is 1 and its height is $\tan h$, so its area is $\frac{1}{2}\tan h$.

Therefore,

$$\frac{\sin h\cos h}{2} < \frac{h}{2} < \frac{\tan h}{2}.$$

Double and divide by $\sin h$ to get

$$\cos h < \frac{h}{\sin h} < \frac{1}{\cos h}$$

then reciprocate to get

$$\frac{1}{\cos h} > \frac{\sin h}{h} > \cos h.$$

Now as $h \to 0$, both ends of this inequality approach 1, so, by the sandwich theorem, so does the middle term. That establishes the limit.

(We're actually using a limit we haven't proven here, namely $\displaystyle\lim_{h\to 0}\cos h = 1$.)

The other limit, $\displaystyle\lim_{h\to 0}\frac{1-\cos h}{h} = 0$. We'll multiply both the numerator and denominator by the conjugate $1+\cos h$.

$$\begin{aligned}\frac{1-\cos h}{h} &= \frac{1-\cos h}{h}\,\frac{1+\cos h}{1+\cos h} = \frac{1-\cos^2 h}{h(1+\cos h)}\\ &= \frac{\sin^2 h}{h(1+\cos h)} = \frac{\sin h}{h}\,\frac{\sin h}{1+\cos h}\end{aligned}$$

Since as $h \to 0$, the first term approaches 1 and the second term approaches $\frac{0}{2} = 0$, therefore the product approaches 0. We've established the limit.

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