CLARK UNIVERSITY

# What are limits?
## Math 120 Calculus I
Fall 2015

**We need limits!** As we've seen, we need limits to define derivatives. We will define the derivative $f'(x)$ of a function $f$ at $x$ by one of the two equivalent limits

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x) = \lim_{b\to x}\frac{f(b)-f(x)}{b-x}.$$

It's time now to formally define limits. Note that in this limit, in fact, in all limits, the variable is never taken to be equal to the constant. That means, for example, in the first limit $\displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ that $h$ doesn't ever take on the value 0, just values near 0 either slightly greater than 0 or slightly less than 0.

Besides using limits for the definition of derivatives, we'll use limits for other purposes, and that means we should develop a general definition for the concept of limit.

In general, we'll have some independent variable, say $x$, approach some constant, say $a$, and ask what some dependent variable, say $y = f(x)$, approaches. If it approaches some constant $L$ as a limit, we'll write

$$\lim_{x\to a} f(x) = L.$$

We'll also express this limit as follows: as $x \to a$, $f(x) \to L$. Read the arrow $\to$ as the word "approaches".

We'll assume that the function $f$ is defined near $a$, but we won't require that $f$ be defined at $a$, in fact, for our application of derivatives, it never will be.

**What doesn't work.** We'll look at the formal definition and analyze it to see what it means. It took mathematicians about 200 years to come up with this definition, from the time that Fermat started to use limits informally to when Cauchy and Weierstrass came up with the definition.

You would think that the definition would say that $\lim_{x\to a} f(x) = L$ means something like "as $x$ is closer to $a$, $f(x)$ gets closer to $L$," but that doesn't work.

*Example 1.* Take this limit for example, $\lim_{x\to 2}(x^2 - 4x + 7)$. As $x$ gets closer to 2, $f(x) = x^2 - 4x + 7$ does actually get closer to 1, but 1 is not the limit. Here's a short table of values of $x$.

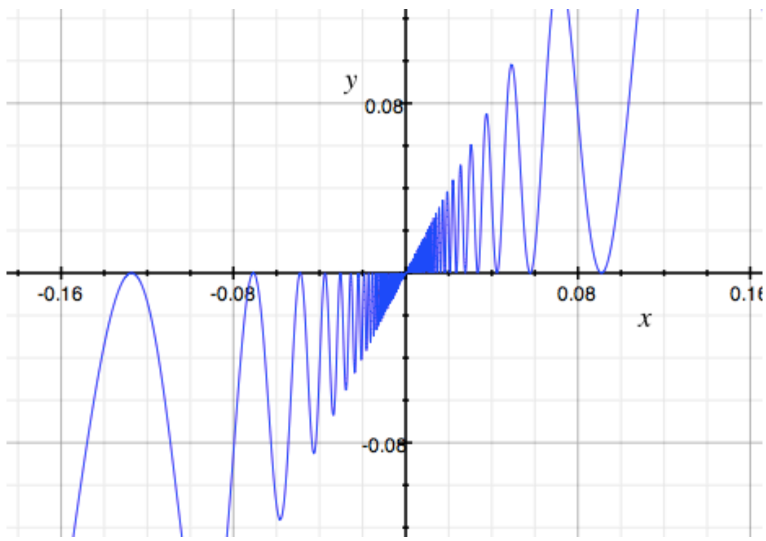
| $x$ | 5 | 4 | 3 | 2.5 | 2.1 | 2.01 |
|---|---|---|---|---|---|---|
| $f(x)$ | 9 | 7 | 4 | 3.25 | 3.01 | 3.001 |

You can see that as you take $x$ closer to 2 that $f(x)$ is always getting closer to 1, but you can also see that the limit is going to be 3, not 1.

Getting closer isn't enough. It's got to be getting arbitrarily close to the limit. The numbers may be getting closer to 1, but not by much. On the other hand, they're getting as close as you like to 3.

*Example 2.* A more subtle problem is that the numbers don't always have to be getting closer to the limit. Here's an example where they get alternately get closer then further from the limit, but eventually get as close as you like to the limit.

$$\lim_{x\to 0} x(1+\sin(1/x))$$




From the graph of $y = x(1+\sin(1/x))$ you can see as $x \to 0$ that $y$ oscillates a lot, but the oscillations keep shrinking as $x$ gets small. $y$ sometimes is exactly 0 but grows up to as much as $2x$ then comes back to 0. By making $x$ sufficiently near 0, you can assure that $y$ is as close to 0 as you like.

**The formal definition.** The formal definition is says something like "you can make $f(x)$ arbitrarily close to $L$ by insisting that $x$ be sufficiently close to $a$."

*Definition.* Assume that a function $f$ is defined near $a$ (but perhaps not defined at $a$). We'll say that $f(x)$ approaches a limit $L$ as $x$ approaches $a$ when it is the case that for each positive value $\epsilon$, there is a positive value $\delta$ (which may depend on $\epsilon$) such that whenever

$$0 < |x-a| < \delta$$

it is the case that

$$|f(x)-L| < \epsilon.$$

That's the definition. The last part is best interpreted as a statement about distances. It says that whenever $x$ is not equal to $a$ but within $\delta$ of $a$, then $f(x)$ is within $\epsilon$ of the limit $L$.

The first part of the definition contains two quantifiers. A quantifier introduces a new variable and indicates the part that variable plays. There are two kinds of quantifiers—universal quantifiers and existential quantifiers—and both appear in this definition. The phrase "for each positive value of $\epsilon$" is a universal quantifier. It indicates that the rest of the statement has to be true for *all* positive $\epsilon$. Universal quantifiers are very common. Every time the word "every", "all", or "each" occurs in English, there's a universal quantifier. There is a special mathematical symbol that is often used to abbreviate universal quantifiers, an upside down A. Here, it would be denoted $\forall\epsilon > 0$.

The phrase "there is a positive value $\delta$" is an existential quantifier. It indicates that there is at least one positive value for $\delta$ that makes the remaining statment true. Existential quantifiers are also very common. Every time "there is", "there exists", "some", "a", or "an" occurs in English, there's an extential quantifier. The symbol used to abbreviate existential quantifiers is an upside down E. $\exists\delta > 0$.

One of the things that makes this definition of limit difficult to understand is the pair of quantifiers. $\forall\epsilon > 0, \exists\delta > 0$. The existential quantifer comes after the universal one, and that means the value of $\delta$ can depend on the value of $\epsilon$. There's actually a third quantifier in the definition. The "whenever" is a universal quantifier for the $x$ that appears in the rest of the definition.

Some adverbs can also indicate quantifiers. When we say there is a $\delta > 0$ so that whenever $x$ is within $\delta$ of $a$, we could also say that whenever $x$ is sufficiently close to $a$. Likewise when we say that for all $\epsilon > 0$ that $f(x)$ is within $\epsilon$ of $L$, we could say that $f(x)$ can be made arbitrarily close to $L$. The words "sufficiently" and "arbitrarily" indicate existential quantifiers and universal quantifiers but don't explicitly mention variables. Using them, we can state the definition in a way that sounds better in English as follows

> $\lim_{x \to a} f(x) = L$ means that by taking $x$ sufficiently close to $a$ (but not equal to $a$), we can make $f(x)$ arbitrarily close to $L$.

We could express this condition even more symbolically as

$$\forall\epsilon > 0, \exists\delta > 0, \forall x \, (0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

Math 120 Home Page at `http://math.clarku.edu/~ma120/`