Math 121 Calculus II
First Test Answers
February 2014


1. [20; 10 points each part] Indefinite integrals and antiderivatives.
   a. A function $f$ has the derivative $f'(x) = 6x^2 - 6x + 4$, and $f(2) = 3$. Determine the function $f$ from that information.

   First integrate $f'$ to determine that $f(x)$ has the form
   
   $$f(x) = 2x^3 - 3x^2 + 4x + C$$

   then use the condition $f(2) = 3$ to determine the value of $C$. Since

   $$f(2) = 2(2^3) - 3(2^2) + 4(2) + C = 12 + C$$

   therefore $C = -9$. Thus, $f(x) = 2x^3 - 3x^2 + 4x - 9$.

   b. Evaluate the indefinite integral

   $$\int (\sec^2 x + 3\tan x) \, dx = \tan x + \frac{1}{3} \sin 3x + C$$

2. [20; 10 points each part] On definite integrals. Evaluate the following integrals. Show your work for credit. (You do not have to find the answer decimally; an unsimplified expression involving numbers is sufficient.)

   a. $\int_0^{\pi/4} \sin^2 x \cos x \, dx$

   The substitution $u = \sin x$ suggests itself since it’s the inner function in the composition $(\sin x)^2$. Then $du = \cos x \, dx$. Note that when $x = 0$, $u = \sin 0 = 0$, and when $x = \pi/4$, $u = \sin \pi/4 = \sqrt{2}/2$. So the integral becomes

   $$\int_0^{\sqrt{2}/2} u^2 \, du = \frac{1}{3} u^3 \bigg|_0^{\sqrt{2}/2}$$

   which equals $\frac{1}{3} (\sqrt{2}/2)^3$.

   b. $\int_3^5 \frac{1}{x + 2} \, dx$

   You may identify an antiderivative right away, but if not, the substitution $u = x + 2$, $du = dx$, will simplify the integral. You’ll get

   $$\int_5^7 \frac{1}{u} \, du = \ln u \bigg|_5^7 = \ln 7 - \ln 5$$

3. [20; 10 points each part] On integrals and derivatives. Let $f(x) = \int_2^x \frac{t^3 + 3}{t^2 + 1} \, dt$. Then $f'(3)$ equals (circle one)

   A) 11/5     B) 6     C) 4/5
   D) 1     E) 0     F) 3

   The derivative of the integral is the integrand, so

   $$f'(x) = \frac{x^3 + 3}{x^2 + 1}.$$  Therefore,

   $$f'(3) = \frac{3^3 + 3}{3^2 + 1} = \frac{30}{10}$$

   Therefore, the answer is F.

   b. In the first semester of calculus, you found derivatives of various functions like

   $$f(x) = \left(\frac{3x}{x^2 + 1}\right)^4.$$  For that one, you used the chain rule and the quotient rule to find that

   $$f'(x) = \frac{324x^3(1 - x^2)}{(x^2 + 1)^5}.$$  Use that result to evaluate the following integral

   $$\int_0^2 \frac{x^3(1 - x^2)}{(x^2 + 1)^5} \, dx$$
Except for the constant factor of 324, the integrand is equal to \( f'(x) \). Therefore, the integral is equal to

\[
\frac{1}{324} f(x) \bigg|_0^2 = \frac{1}{324} (f(2) - f(0)) = \frac{1}{324} \left( \frac{6}{5} \right)^4
\]

4. [10] **On areas between curves.** Write down an integral which gives area of the finite region bounded between \( y = 4 - x^2 \) and \( y = 4 - 2x \) for \( 0 \leq x \leq 2 \), sketched below. Do not compute the value of the integral.

![Graph of curves](image)

The area \( A \) is the integral of the cross sectional length. That length at \( x \) is \((4 - x^2) - (4 - 2x) = 2x - x^2\). Therefore, the area is

\[
A = \int_0^4 (2x - x^2) \, dx.
\]

5. [10] **On volumes of solids of revolution.** The region described in 4 is rotated about the \( x \)-axis to create a solid of revolution. Write down an integral which gives the volume of that solid. Do not compute the value of the integral.

The volume \( V \) is the integral of the cross sectional area \( A(x) \). The cross section at \( x \) is the “washer” whose inner radius is \( 4 - 2x \) and whose outer radius is \( 4 - x^2 \). Therefore,

\[
A(x) = \pi (4 - x^2)^2 - \pi (4 - 2x)^2.
\]

Thus, the volume is

\[
V = \int_0^2 \left( \pi (4 - x^2)^2 - \pi (4 - 2x)^2 \right) \, dx
\]

which, of course, can be simplified.

Note that the integral \( \int_0^2 (\pi (4 - x^2)^2 - \pi (4 - 2x)^2) \, dx \) describes the volume of a different solid entirely.

6. [20; 10 points each part] **On arc lengths and surfaces of revolution.** Consider the curve given by the equation \( y = \ln x \) between \( x = 1 \) and \( x = e \).

a. Write down an integral which gives the length of this curve. Do not evaluate the integral.

The arclength is

\[
\int_a^b ds = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx
\]

Here, \( \frac{dy}{dx} = 1/x \), so the integral is

\[
\int_1^3 \sqrt{1 + 1/x^2} \, dx
\]

b. That curve is rotated around the \( x \)-axis to form a surface of revolution. Write down an integral which gives the area of that surface. Do not compute the value of the integral.

The surface area \( A \) is \( \int_a^b 2\pi y \, ds \) where \( ds \) is as in part a, and \( y = \ln x \). Therefore,

\[
A = \int_1^e 2\pi (\ln x) \sqrt{1 + 1/x^2} \, dx.
\]