CLARK UNIVERSITY 1887

Final Exam
Math 130 Linear Algebra
D Joyce, December 2013

1. [18; 6 points each part] Consider the linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ such that $T(\mathbf{e}_1) = \mathbf{e}_2$, $T(\mathbf{e}_2) = \mathbf{e}_3$, and $T(\mathbf{e}_3) = \mathbf{e}_1$.

a. What is the 3×3 matrix A that represents T, that is, $T(\mathbf{x}) = A\mathbf{x}$?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

b. The matrix A^3 represents $T \circ T \circ T$, the triple composition of T. What matrix is that?

It's the identity matrix.

c. Is the transformation T a reflection, and if so, across which plane, or is it a rotation, and if so, then by what angle and about what axis, or is it some other transformation of space?

It's a rotation by 120° about the line $x = y = z$.

2. [18; 6 points each part] Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

a. Find the characteristic polynomial of A.

It's the determinant of $A - \lambda I$. Here it is computed in a partially factored form in preparation for part **b**.

$$\begin{aligned} & \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{vmatrix} \\ = \ & (3-\lambda)((2-\lambda)^2 - 1) \\ = \ & -(\lambda+2)(3 - 4\lambda + \lambda^2) \end{aligned}$$

b. Find the eigenvalues for A.

The characteristic polynomial factors as $(3-\lambda)^2(1-\lambda)$, so there are two eigenvalues. $\lambda_1 = 3$ is a double root, and $\lambda_2 = 1$ is a single root.

c. Choose one of the eigenvalues that you found in part **b** and determine all the eigenvectors for it.

If you chose the eigenvalue $\lambda_1 = 3$, you'd solve the equation $(A - 3I)\mathbf{x} = 0$. Since $A - 3I$ is the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix},$$

the solutions are of the form $\mathbf{x} = (x, z, z)$ where x and z are any real numbers.

If you chose the eigenvalue $\lambda_2 = 1$, you'd solve the equation $(A-)\mathbf{x} = 0$. Since $A - 3I$ is the matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

the solutions are of the form $\mathbf{x} = (0, -z, z)$ where z is any real number.

3. [10] Find a basis and the dimension of the vector space of all vectors of the form (a, b, c, d) in $\mathbf{R}^4$, where $a + b = c + d$.

The solution space consists of vectors of the form $(c + d - b, b, c, d)$ where b, c, and d can be any numbers. Its dimension is 3 and a basis is $\{(-1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1)\}$.

4. [12] Prove that if A is an upper triangular matrix ,then the eigenvalues of A are the elements on the main diagonal of A.

The eigenvalues are the roots of the characteristic polynomial, which is $f(\lambda) = \det(A - \lambda I)$. Since A is an upper triangular matrix, so is $A - \lambda I$, so its determinant is the product of its diagonal elements. Thus

$$f(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

The roots of $f(\lambda)$ are therefore $a_{11}, a_{22}, \ldots, a_{nn}$, the elements on the main diagonal of A. Q.E.D.

5. [10] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}.$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3 & 3 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & -1 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & -1 & 1 & -1 \\ 0 & 1 & 2 & -2 & 1 & -1 \\ 0 & 0 & -3 & 4 & -2 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & -1 & 1 & -1 \\ 0 & 1 & 2 & -2 & 1 & -1 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 \end{array}\right]$$
$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & \frac{5}{3} & -\frac{1}{3} & 1 \\ 0 & 1 & 2 & \frac{2}{3} & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & -\frac{4}{3} & \frac{2}{3} & -1 \end{array}\right]$$

The right half of the last matrix is A^{-1}.

6. [12] Consider the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^3$ represented by the matrix

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 2 & 1 & 0 & 3 & 2 \\ 4 & 5 & 6 & 1 & 2 \end{bmatrix}.$$

Determine the rank and nullity of T. (Show your work)

First row reduce the matrix. Pivot on the 1 in the upper left corner to clear the elements under it in the first column. Then pivot on the second element in the second row to clear the element below it. You'll get

$$\begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 0 & -3 & -6 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are 2 nonzero rows after that, so the rank is 2. By the dimension theorem, the rank plus the nullity equals the dimension of the domain, which in this case is 5. Therefore the nullity is 3. (You could also directly find the nullity by determining the dimension of the solution space.)

7. [21; 3 points each] True/false.

a. Given a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, there is a basis of $\mathbf{R}^n$ whose basis vectors are all eigenvectors. *False.* Some do, some don't.

b. Inner products distribute over addition, that is, $\langle \mathbf{u} + \mathbf{v} | \mathbf{w} \rangle = \langle \mathbf{u} | \mathbf{w} \rangle + \langle \mathbf{v} | \mathbf{w} \rangle$. *True.* Inner products are linear.

c. The determinant of a square matrix is equal to the determinant of its transpose. *True.* That's one of the amazing things about determinants.

d. The inequality $\|\mathbf{w} - \mathbf{v}\| \leq \|\mathbf{w}\| + \|\mathbf{v}\|$ is known as Cramer's inequality. *False.* That's the triangle inequality. Does Cramer even have an inequality named after him?

e. Matrix multiplication is commutative: $AB = BA$. *False.* Rarely does matrix multiplication commute.

f. The following set S is a basis for $\mathbf{R}^6$. $S = \{(3,2,0,8,-5,2), (4,3,-2,0,4,1), (-3,2,1,4,5,2), (2,3,-2,1,0,0), (0,3,2,3,2,1)\}$

False. There aren't enough vectors to be a basis.

g. The fixed points $\mathbf{x}$ of a matrix transformation $T(\mathbf{x}) = A\mathbf{x}$ are eigenvectors with eigenvalue 1. *True.* Both mean $T(\mathbf{x}) = 1\mathbf{x}$.