First Test Answers
Math 130 Linear Algebra

D Joyce, October 2013

Scale. 85–100 A, 73–84 B, 54–72 C. Median 75 (B-).

1. [10] The following matrix describes a system of linear equations in five unknowns—v, w, x, y, z. What is the general solution to this system?

$$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & 2 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \end{array}\right]$$

You can do a couple row reductions to put it in the form

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -36 & -32 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \end{array}\right]$$

And from that you can see what v, w, and y are in terms of x and z, namely,

$$\begin{array}{rcl} v & = & -32 + 36z \\ w & = & 8 - 4z \\ y & = & -2 - 7z \end{array}$$

Therefore, the general solution is $(v, , w, x, y, z) =$

$$(-32 + 36z, 8 - 4z, x, -2 - 7z, z)$$

where x and z can be any numbers.

2. [10] Explain in a sentence or two why it is the case that if a subset S of a vector space V spans V, then any subset T of V that contains S (that is, $S \subseteq T$) also spans V.

Since S spans V, every vector in V is a linear combination of vectors in S, but every vector in S is a vector in T, so every vector in V is a linear combination of vectors in T.

3. [20] Determine if the set

$$S = \{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$$

of three vectors in $\mathbf{R}^3$ is independent or dependent. Show your work for credit.

Is there a nontrivial solution to the vector equation

$$x(1, -1, 2) + y(1, -2, 1) + z(1, 1, 4) = (0, 0, 0)?$$

Solve the system of homogeneous linear equations

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 0 \\ -x & & -2y & + & z & = & 0 \\ 2x & + & y & + & 4z & = & 0 \end{array}$$

Solve it however you like to see if there are any nontrivial solutions. Here I'll write down the coefficient matrix and use that.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

Note that the original three vectors are the columns of this matrix. Row reduce it to get in echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now the third row is all zeros, and that means there will be a nontrivial solution. When a system of homogeneous linear equations has more variables (3 in this case) than independent equations (2 in this case), it always has a nontrivial solution. Therefore the set is linearly dependent.

It isn't necessary to find the solution set to answer the question, but if you did, you'd find the solutions are of the form $(x, y, z) = (-3z, 2z, z)$ where z can be any number. In particular, one nontrivial solution is $(x, y, z) = (-3, 2, 1)$.

4. [12] Recall that a function $f : \mathbf{R} \to \mathbf{R}$ is an *odd function* if $f(-x) = -f(x)$ for each real number x. (Examples are x^3, x^5, and $\sin x$.) Determine whether or not the set of odd functions is a subspace of the set of all functions $f : \mathbf{R} \to \mathbf{R}$. After you've made your determination, write one sentence stating whether or not it is a subspace and how you made your determination.

The constant function 0 is an odd function, and odd functions are closed under addition and scalar multiplication. Therefore the set of odd functions form a subspace of all functions.

5. On abstract vector spaces [24]. Using only the additive axioms of vector spaces prove that given two vectors $\mathbf{v}$ and $\mathbf{w}$ in the vector space

$$\mathbf{v} + \mathbf{w} = \mathbf{0} \text{ if and only if } \mathbf{w} = -\mathbf{v}.$$

The four additive axioms are listed below.

This is a bi-implication, so there are two parts to the proof. There are many possible proofs. Here's one.

Proof $\Rightarrow$: Suppose that $\mathbf{v} + \mathbf{w} = \mathbf{0}$. Add $-\mathbf{v}$ to each side, which exists by axiom 4. Then

$$(-\mathbf{v}) + (\mathbf{v} + \mathbf{w}) = (-\mathbf{v}) + \mathbf{0}.$$

By axiom 2, the left side equals $((-\mathbf{v}) + \mathbf{v}) + \mathbf{w}$, and by axiom 1 that equals $(\mathbf{v} + (-\mathbf{v})) + \mathbf{w}$, which by axiom 4 equals $\mathbf{0} + \mathbf{w}$, and by axiom 3, that equals $\mathbf{w}$. But by axiom 3, the right side equals $-\mathbf{v}$. Therefore,**w**=-**v**. Q.E.D.

Proof $\Leftarrow$: Suppose that $\mathbf{w} = -\mathbf{v}$. Then the statement $\mathbf{v}+\mathbf{w} = \mathbf{0}$ directly follows since it simply says that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$, which is axiom 4. Q.E.D.

1. Vector addition is commutative: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ for all vectors $\mathbf{v}$ and $\mathbf{w}$;

2. Vector addition is associative: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$;

3. There is a vector, denoted $\mathbf{0}$ and called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$ for each vector $\mathbf{v}$;

4. For each vector $\mathbf{v}$, there is another vector, denoted $-\mathbf{v}$ and called the *negation* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.

6. [24; 4 points each] True/false.

a. Subsets of linearly dependent sets are linearly dependent. *False.* For example, the set $\{(1,0,),(0,1),(1,1)\}$ is linearly dependent, but its subset $\{(1,0,),(0,1)\}$ is linearly independent.

b. Every system of linear equations has a solution. *False.* Some are inconsistent. For example the pair of equations $x + y = 2$ and $x + y = 3$ has no solution

c. The intersection of any two subspaces of a vector space is also subspace of that vector space. *True.* If each has $\mathbf{0}$ and is closed under addition and scalar multiplication, then so is their intersection.

d. If three nonzero vectors are linearly dependent, then one of them is a scalar multiple of one of the others. *False.* For example, the set $\{(1,0,),(0,1),(1,1)\}$ is linearly dependent but no one of the three vectors is a scalar multiple of another.

e. The set $S = \{(v_1, v_2, v_3) \,|\, 3v_1 + 2v_2 - v_3 = 1 \text{ and } v_1 = v_2 + 4v_3 = 0\}$ is a subspace of $\mathbf{R}^3$. *False.* It doesn't include $(0,0,0)$. It's also not closed under addition or scalar multiplication.

f. The set $S = \{ax^2 + bx + c \,|\, a, b, c \in \mathbf{R}\text{and } a \neq 0\}$ of quadratic polynomials with real coefficients is a vector space. *False.* It's not closed under addition. Although $2x^2 + 2x$ and $-2x^2 - 5x$ are both quadratic, their sum $-3x$ is not.