Math 130 Linear Algebra
Second Test
November 2013

You may use a calculator and a sheet of notes. Leave your answers as expressions such as $e^{2\sqrt{\sin^2(\pi/6)}}$ if you like. Show all your work for credit. Points for each problem are in square brackets.

1. [18] Give examples of transformations with the following properties. You don’t have to prove that they have the properties. Just specify the transformations.

   a. [5] A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose rank is 0 and nullity is 2.

   b. [5] A linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose rank is 1 and nullity is 1.

   c. [8] Two different linear transformations $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ both with rank 2 and nullity 0.
2. [20] The set of five vectors

\[ S = \{(1, -3, -2), (1, 2, 2), (3, 1, 2), (0, 5, 4), (2, -1, 1)\} \]

spans \( \mathbb{R}^3 \). Find a subset of \( S \) which is a basis for \( \mathbb{R}^3 \). Show your work.
3. [28; 4 points each] True/false. For each sentence write the whole word “true” or the whole word “false”. If it’s not clear whether it should be considered true or false, you may explain in a sentence if you like.

_________ a. If $T : V \rightarrow W$ is a linear transformation, and if the vectors $T(v_1), \ldots, T(v_k)$ are linearly independent in $W$, then the vectors $v_1, \ldots, v_k$ are linearly independent in $V$.

_________ b. If a linear transformation $T : V \rightarrow W$ goes between two vector spaces $V$ and $W$ of the same dimension, then it’s an isomorphism.

_________ c. If the rank of a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is $m$, then $T$ is surjective (that is to say, $T$ is onto).

_________ d. Given two $n \times n$ matrices $A$ and $B$, if $(A + B)(A - B) = A^2 - B^2$, then $AB = BA$.

_________ e. Given two $n \times n$ matrices $A$ and $B$, if both $A$ and $B$ are invertible, then so is $A + B$.

_________ f. The vector space $M_{2 \times 6}$ of $2 \times 6$ matrices is isomorphic to the vector space $M_{3 \times 4}$ of $3 \times 4$ matrices.

_________ g. If a set $S$ spans a vector space $V$, then every every vector in $V$ can be written as a linear combination of vectors from $S$ in only one way.
4. [20] Consider the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ described by the matrix $A$, that is, $T(\mathbf{x})$ is found by evaluating $A$ times the column matrix $\mathbf{x}$, where

$$A = \begin{bmatrix}
1 & 2 & 0 & 3 & 1 \\
0 & 0 & 1 & 4 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

a. [7] Determine a basis for the kernel of $T$. (This should be a set of vectors in $\mathbb{R}^5$.)

b. [3] What is the nullity of $T$?

c. [7] Determine a basis for the image of $T$. (This should be a set of vectors in $\mathbb{R}^3$.)

d. [3] What is the rank of $T$?
5. [15] If $A$ is a $5 \times 3$ matrix, prove that the rows of $A$ are linearly dependent. (There are several ways that you can approach this. Be sure to write a clear and complete explanation.)