CLARK UNIVERSITY 1887

Second Test Answers
Math 130 Linear Algebra
D Joyce, November 2013

Scale. 85–100 A, 70–84 B, 50–69 C. Median 65 (C+).

1. [18] Give examples of transformations with the following properties. You don't have to prove that they have the properties. Just specify the transformations.

a. [5] A linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ whose rank is 0 and nullity is 2.

The only such transformation is the constantly $\mathbf{0}$ transformation $(x, y) \mapsto (0, 0)$. If you prefer, you could exhibit this transformation as the 2×2 matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

b. [5] A linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ whose rank is 1 and nullity is 1.

There are many of these. They're called projections, and the simplest one is $(x, y) \mapsto (x, 0)$, the projection to the x-axis. This projection has the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

c. [8] Two different linear transformations $\mathbf{R}^2 \to \mathbf{R}^2$ both with rank 2 and nullity 0.

Examples of these are reflections like $(x, y) \mapsto (x, -y)$, rotations like $(x, y) \mapsto (-y, x)$, expansions like $(x, y) \mapsto (2x, 2y)$, and lots of other transformations. In fact, if you choose constants a, b, c, and d at random, then in all probability the transformation $(x, y) \mapsto (ax + by, cx + dy)$, described by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, will have rank 2 and nullity 0.

2. [20] The set of five vectors $S =$

$$\{(1, -3, -2), (1, 2, 2), (3, 1, 2), (0, 5, 4), (2, -1, 1)\}$$

spans $\mathbf{R}^3$. Find a subset of S which is a basis for $\mathbf{R}^3$. Show your work.

There are several ways you can find a linearly independent subset of three elements of S. Using only the definition of linear independence, you could search through the vectors adding independent vectors one at a time until you have enough. The first two are clearly independent since neither is a multiple of the other. So it's a matter of finding one of the remaining three that is independent of these first two. Although the third and fourth aren't independent of them, the last one is.

In fact, for this set S of vectors, the first four lie in a plane in $\mathbf{R}^3$ but the last one doesn't. Every answer includes the last vector and any two of the first four vectors.

3. [28; 4 points each] True/false.

a. If $T : V \to W$ is a linear transformation, and if the vectors $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ are linearly independent in W, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent in V. *True.* If there were a linear dependence among $\mathbf{v}_1, \ldots, \mathbf{v}_k$, then T would preserve that linear dependence giving a linear dependence among $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$.

b. If a linear transformation $T : V \to W$ goes between two vector spaces V and W of the same dimension, then it's an isomorphism. *False.* Although V and W would be isomorphic if they had the same dimension, it needn't be T that gives the isomorphism. For example $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(x, y) = (x + y, 0)$ is a linear transformation between vector spaces of the same dimension, but it's not an isomorphism.

c. If the rank of a linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ is m, then T is surjective (that is to say, T is onto). *True.* If the rank of T is m, then the dimension of the image of T is m, and since $\mathbf{R}^m$ has dimension m, therefore $\mathbf{R}^m$ is the image of T, that is, T is surjective.

d. Given two $n \times n$ matrices A and B, if

$$(A + B)(A - B) = A^2 - B^2,$$

then $AB = BA$. *True.* Since $(A+B)(A-B)$ always equals $A^2 + AB - BA - B^2$, if that equals $A^2 - B^2$, then $AB - BA = 0$, so $AB = BA$.

e. Given two $n \times n$ matrices A and B, if both A and B are invertible, then so is $A + B$. *False.* The product AB would be invertible, but the sum $A + B$ needn't be. For example, when $A = I$ and $B = -I$, both A and B are invertible, but their sum, 0, is not.

f. The vector space $M_{2\times6}$ of 2×6 matrices is isomorphic to the vector space $M_{3\times4}$ of 3×4 matrices. *True.* Two vector spaces are isomorphic if and only if they have the same dimension, and these two vector spaces both have dimension 12.

g. If a set S spans a vector space V, then every every vector in V can be written as a linear combination of vectors from S in only one way. *False.* If S spans V, then then every vector can be written as a linear combination in *at least one* way, but unless S is also linearly independent, vectors can be written in more than one way.

4. [20] Consider the linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^3$ described by the matrix A, that is, $T(\mathbf{x})$ is found by evaluating A times the column matrix $\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a. [7] Determine a basis for the kernel of T. (This should be a set of vectors in $\mathbf{R}^5$.)

The kernel consists of vectors $(x_1, x_2, x_3, x_4, x_5)$ (which you can exhibit as 5×1 column matrices if you prefer) such that $x_1 + 2x_2 + 3x_4 + x_5 = 0$ and $x_3 + 4x_4 + 2x_5 = 0$. The first equation determines what x_1 has to be in terms of x_2, x_4, and x_5, while the second equation determines what x_3 has to be in terms of x_4 and x_5. Thus, the general solution is of the form $(x_1, x_2, x_3, x_4, x_5) =$

$$(-2x_2 - 3x_4 - x_5, x_2, -4x_4 - 2x_5, x_4, x_5)$$

where x_2, x_4, and x_5 may be any numbers.

A basis for the kernel can be made from the three ways that one of x_2, x_4, and x_5 is set to 1 while the other two are set to 0. So one basis is

$$\beta = \{(-2, 1, 0, 0, 0), (-3, 0, -4, 1, 0), (-1, 0, -2, 0, 1)\}.$$

There are, of course, many other bases, but every basis has 3 elements.

b. [3] What is the nullity of T?

It's 3, the number of vectors in the basis β.

c. [7] Determine a basis for the image of T. (This should be a set of vectors in $\mathbf{R}^3$.)

The image is spanned by the columns of the matrix A. Reading those columns as triples, we see the image is spanned by

$$(1, 0, 0), (2, 0, 0), (0, 1, 0), (3, 4, 0), (1, 2, 0).$$

There's a lot of redundancy in this spanning set. It can be reduced to a basis by taking just the first and third elements $\gamma = \{(1, 0, 0), (0, 1, 0)\}$.

d. [3] What is the rank of T?

It's 2, the number of elements in the basis γ. Also, we know the rank is two since the dimension of the domain is 5, the dimension of the kernel is 3, so the dimension of the image is $5 - 3 = 2$.

5. [15] If A is a 5×3 matrix, prove that the rows of A are linearly dependent. (There are several ways that you can approach this. Be sure to write a clear and complete explanation.)

Here's one argument based directly on the dimension of $\mathbf{R}^3$. There are five rows in A, each one being a vector in $\mathbf{R}^3$. Since the dimension of $\mathbf{R}^3$ is 3, therefore at most 3 vectors among these 5 can be independent. Thus, the set of all five of them is linearly dependent set.

Here's a different argument based on ranks of a matrix. The column space is spanned by the three columns, so it has at most dimension 3. Therefore the rank of A is at most 3. Therefore, the row space has at most dimension 3. Since there are 5 rows, they can't be independent since the row space can't be as high as 5.