CLARK UNIVERSITY

Change of coordinates
Math 130 Linear Algebra
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The coordinates of a vector $\mathbf{v}$ in a vector space V with respect to a basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{v}_b\}$ are those coefficients c_i which uniquely express $\mathbf{v}$ as as linear combination of the basis vectors

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n.$$

These coefficients $v_1, v_2, \ldots, v_n$ are called *coordinates* with respect to the basis β. The column vector of these coordinates is denoted $[\mathbf{v}]_\beta$.

$$[\mathbf{v}]_\beta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

When the basis is the standard basis for F^n

$$\epsilon = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\},$$

then the coordinates $[\mathbf{v}]_\epsilon$ of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are just the usual coordinates of $\mathbf{v}$.

$$[\mathbf{v}]_\epsilon = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Changing between standard coordinates with respect to another. Take the case when V is F^n and the basis β is not the standard basis ϵ. We may have the standard coordinates of a vector and want the β coordinates of it, or vice versa. How do we convert back and forth?

There's a transition matrix for that. Let the β-basis vector $\mathbf{b}_j$ have standard coordinates $\mathbf{b}_j = (b_{1j}, b_{2j}, \ldots, b_{nj})$, so

$$[\mathbf{b}_j]_\epsilon = \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

Collect these in the columns of a matrix $P_{\epsilon\leftarrow\beta}$ to form a *transition* matrix.

$$P_{\epsilon\leftarrow\beta} = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{bmatrix}$$

Now, the β-coordinates for $\mathbf{b}_j$ are all 0s except a 1 in the j^{th} coordinate, so multiplying $P_{\epsilon\leftarrow\beta}$ by the column matrix $[\mathbf{b}_j]_\beta$ picks out the j^{th} column, which are the standard coordinates for $\mathbf{b}_j$, so $P_{\epsilon\leftarrow\beta}[\mathbf{b}_j]_\beta = [\mathbf{b}_j]_\epsilon$.

More generally, for an arbitrary vector $\mathbf{v}$ in F_n, the β-coordinates $[\mathbf{v}]_\beta$ of $\mathbf{v}$ as a linear combination of the basis vectors in β, so

$$P_{\epsilon\leftarrow\beta}\,[\mathbf{v}]_\beta = [\mathbf{v}]_\epsilon.$$

Thus, the transition matrix $P_{\epsilon\leftarrow\beta}$ converts from β coordinates to ϵ coordinates.

Unfortunately, it's usually the reverse change of coordinates that we want. But we can do that, too.

To convert the other way, just invert the matrix $P_{\epsilon\leftarrow\beta}$ to get

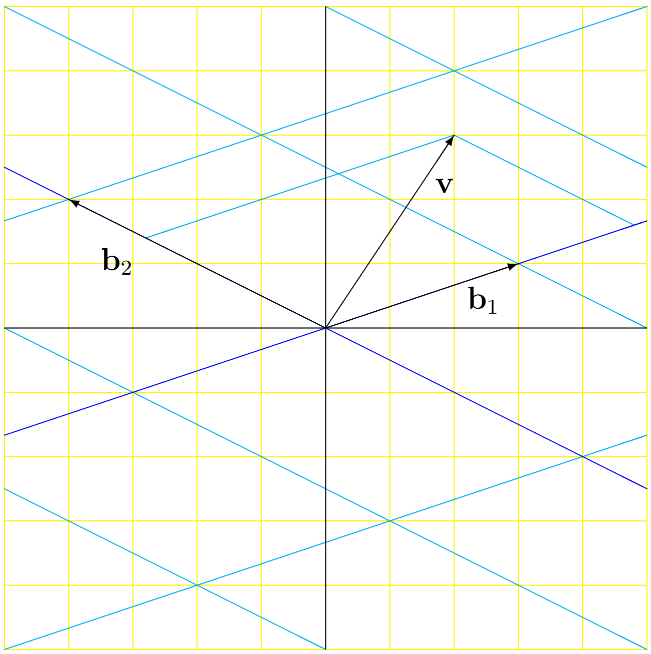
$$P_{\beta\leftarrow\epsilon} = (P_{\epsilon\leftarrow\beta})^{-1}.$$

Then,

$$P_{\beta\leftarrow\epsilon}\,[\mathbf{v}]_\epsilon = [\mathbf{v}]_\beta.$$

Example 1. A low-dimensional example will help explain things. Let $\beta = \{\mathbf{b}_1, \mathbf{b}_2\}$ be a basis of $\mathbf{R}^2$ where $\mathbf{b}_1 = (3, 1)$ and $\mathbf{b}_2 = (-4, 2)$.

In the figure, the standard coordinates are shown with black axes and a yellow grid, while the β-coordinates are shown with blue axes and a cyan grid.



The transition matrix is

$$P_{\epsilon\leftarrow\beta} = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}$$

and its inverse is

$$P_{\beta\leftarrow\epsilon} = (P_{\epsilon\leftarrow\beta})^{-1} = \begin{bmatrix} 0.2 & 0.4 \\ -0.1 & 0.3 \end{bmatrix}$$

Take a typical vector $\mathbf{v}$, say $\mathbf{v} = (2, 3)$. Then its β-coordinates are

$$[\mathbf{v}]_\beta = P_{\beta\leftarrow\epsilon}[\mathbf{v}]_\epsilon = \begin{bmatrix} 0.2 & 0.4 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 0.7 \end{bmatrix}$$

The same vector $\mathbf{v}$ in the plane can be described in two ways. In standard coordinates go 2 $\mathbf{e}_1$'s (2 units right) and 3 $\mathbf{e}_2$'s (3 units up), or in β-coordinates, go 1.6 $\mathbf{b}_1$'s and 0.7 $\mathbf{b}_2$'s.

Example 2. A three-dimensional case. Let $V = \mathbf{R}^3$, and consider the basis $\beta = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ where $\mathbf{b}_1 = (-1, 1, 1)$, $\mathbf{b}_2 = (1, -1, 1)$, and $\mathbf{b}_3 = (1, 1, -1)$. The transition matrix which converts β-coordinates to standard ϵ-coordinates puts the $\mathbf{b}_j$'s in columns

$$P_{\epsilon\leftarrow\beta} = \left[\begin{array}{c|c|c} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right]$$

Suppose we had the β-coordinates for a vector, say the vector $\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2 + 4\mathbf{b}_3$. Its β-coordinates are $[\mathbf{v}]_\beta = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. We can use $P_{\epsilon\leftarrow\beta}$ to convert those to standard ϵ-coordinates:

$$\begin{aligned} [\mathbf{v}]_\epsilon &= P_{\epsilon\leftarrow\beta}[\mathbf{v}]_\beta \\ &= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \end{aligned}$$

Therefore, as a 3-tuple, $\mathbf{v}$ is $(5, 3, 1)$.

The inverse of the matrix $P_{\epsilon\leftarrow\beta}$ is the matrix $P_{\beta\leftarrow\epsilon}$, and it can be computed by the methods described before. You'll find

$$P_{\beta\leftarrow\epsilon} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Using that matrix we can convert from standard ϵ-coordinates to β-coordinates. For example, if we take standard coordinates for the vector $\mathbf{v}$ we had before, $[\mathbf{v}]_\epsilon = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$, and multiply on the left by $P_{\beta\leftarrow\epsilon}$, we should get the β-coordinates we started with

$$\begin{aligned} [\mathbf{v}]_\beta &= P_{\beta\leftarrow\epsilon}[\mathbf{v}]_\epsilon \\ &= \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \end{aligned}$$

Changing between coordinates with respect to two different bases. How do you convert between coordinates $[\mathbf{v}]_\gamma$ of a vector $\mathbf{v}$ with respect to a basis γ and coordinates $[\mathbf{v}]_\beta$ with respect to a different basis β?

One way is to do the same as we just did where the basis γ replaces the standard basis ϵ. Start by finding the coordinates of the basis vectors of β with respect to the basis γ. Then put them in columns in a matrix, which we denote $P_{\gamma\leftarrow\beta}$. And we get analogous results:

$$[\mathbf{v}]_\gamma = P_{\gamma\leftarrow\beta}\,[\mathbf{v}]_\beta$$

but for two different bases γ and β. If you want the reverse change of coordinates, invert the matrix.

$$P_{\beta\leftarrow\gamma} = (P_{\gamma\leftarrow\beta})^{-1}.$$

Then,

$$P_{\beta\leftarrow\gamma}\,[\mathbf{v}]_\gamma = [\mathbf{v}]_\beta.$$

In the case that the vector space V is F^n, we can use the standard basis as an intermediate step. We can also compute $P_{\gamma\leftarrow\beta}$ as a composition

$$P_{\gamma\leftarrow\beta} = P_{\gamma\leftarrow\epsilon}\,P_{\epsilon\leftarrow\beta} = (P_{\epsilon\leftarrow\gamma})^{-1}P_{\epsilon\leftarrow\beta}$$

which is easy to use since the columns of $P_{\epsilon\leftarrow\gamma}$ and $P_{\epsilon\leftarrow\beta}$ are the basis vectors of γ and of β, respectively.

Matrix representation of linear operators. A linear operator is just a linear transformation $T : V \to V$ from a vector space to itself. In order to represent a linear transformation between two different vector spaces, you need to choose a basis for each, but for linear operators, only one basis β for V is needed. If you choose a different basis γ for V, you'll get a different matrix.

If the matrix $[T]_\beta^\beta$ represents T for the basis β, how can you find the matrix $[T]_\gamma^\gamma$ that represents T for the basis γ? Answer, just use the two change of coordinate matrices $P_{\gamma\leftarrow\beta}$ and $P_{\gamma\leftarrow\beta}$.

If you start with a vector $[\mathbf{v}]_\gamma$ in γ-coordinates, first hit it with $P_{\beta\leftarrow\gamma}$ to get it in β-coordinates. Now you've got $[\mathbf{v}]_\beta$, so hit that with $[T]_\beta^\beta$ to get $[T(\mathbf{v})]_\beta$. Finally, hit that with $P_{\gamma\leftarrow\beta}$ to get $[T(\mathbf{v})]_\gamma$. Thus,

$$[T]_\gamma^\gamma = P_{\gamma\leftarrow\beta}[T]_\beta^\beta P_{\beta\leftarrow\gamma}.$$

Since $(P_{\gamma\leftarrow\beta}) = P_{\beta\leftarrow\gamma}^{-1}$, we can also write that equation as

$$[T]_\gamma^\gamma = P_{\beta\leftarrow\gamma}^{-1}[T]_\beta^\beta P_{\beta\leftarrow\gamma}.$$

This observation yields the following theorem where the matrix Q in the statement is the transition matrix $P_{\beta\leftarrow\gamma}[T]$.

Theorem 3. Two matrices A and B represent the same linear operator if and only if there is an invertible matrix Q such that

$$B = Q^{-1}AQ.$$

Similar matrices and equivalence relations. Knowing when two matrices represent the same linear operator is so important that there's a name for them.

Definition 4. Two square matrices A and B are said to be *similar* or *conjugate* when there is an invertible matrix Q such that $B = Q^{-1}AQ$. We'll denote similar matrices $A \sim B$.

With that definition, we can summarize the previous theorem as saying similar matrices represent the same linear operator.

Similarity is a binary relation that has three important properties

- Reflexivity. Any square matrix is similar to itself. $A \sim A$.
- Symmetry. If one matrix is similar to another, then the other is similar to it. $A \sim B$ implies $B \sim A$.
- Transitivity. If one matrix is similar to another, and the second is similar to the third, then the first is similar to the third. $A \sim B$ and $B \sim C$ imply $A \sim C$.

Any binary relation that has these three properties is called an *equivalence relation*.

Equivalence relations occur throughout mathematics. You're familiar with a few of them. For example, similarity of triangles in geometry. Also congruence of triangles. In calculus, having the same derivative is an equivalence relation although it's usually not called an equivalence relation in a calculus course. In number theory, congruence modulo n is an equivalence relation.

Outside of mathematics equivalence relations are common, too. Being the same height, being on the same basketball team in a sports league, and having the same parents are three different equivalence relations.

In fact, the word "same" indicates there's an associated equivalence relation.

Math 130 Home Page at
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