CLARK UNIVERSITY

Coordinates
Math 130 Linear Algebra

D Joyce, Fall 2015

Throughout this discussion, F refers to a fixed field. In application, F will usually be $\mathbf{R}$.

We've seen how each linear transformation $T : F^n \to F^m$ corresponds to an $m \times n$ matrix A. The matrix A is the standard matrix representing T and its j^{th} column consists of the coordinates of the vector $T(\mathbf{e}_j)$ where $\mathbf{e}_j$ is the j^{th} vector in the standard basis ϵ of F^n.

We'll use this connection as we derive algebraic operations on linear transformations to translate those operations to algebraic operations on matrices.

Before we do that, however, we should see that we can use use matrices to represent linear transformations between other vector spaces so long as we've got coordinates for them.

Extending the standard matrix to transformations $T : V \to W$. Coordinates on vector spaces are determined by bases for those spaces.

Recall that a basis $\beta = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ of a vector space V sets up a coordinate system on V. Each vector $\mathbf{v}$ in V can be expressed uniquely as a linear combination of the basis vectors

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n.$$

The coefficients in that linear combination form the coordinate vector $[\mathbf{v}]_\beta$ relative to β

$$[\mathbf{v}]_\beta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The function $\phi_\beta : V \to F^n$ which sends $\mathbf{v}$ to $[\mathbf{v}]_\beta$ is an isomorphism.

Although an isomorphism doesn't mean that V is identical to F^n, it does mean that coordinates work exactly the same.

One place where this can be confusing is when V is F^n, but the basis β is not the standard basis. Having two bases for the same vector space is confusing, but sometimes useful.

Matrices for transformations $V \to W$. So far, we can represent a transformation $T : F^n \to F^m$ by an $m \times n$ matrix A. The entries for A were determined by what T did to the standard basis of F^n. The coordinates of $T(\mathbf{e}_j)$ were placed in the j^{th} column of A. It followed that for any vector $\mathbf{v}$ in F^n, the i^{th} coordinate of it's image $\mathbf{w} = T(\mathbf{v})$ was

$$w_i = A_{i1}v_1 + A_{i2}v_2 + \cdots + A_{ij}v_n = \sum_{j=1}^{n} A_{ij}v_j$$

We can represent a transformation $T : V \to W$ by a matrix as well, but it will have to be relative to one basis β for V and another basis γ for W. Suppose that $\beta = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ is an ordered basis for V and $\gamma = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_m)$ is an ordered basis for W. We'll put the γ-coordinates of the $T(\mathbf{b}_j)$ in the j^{th} column of A, just like we did before. If we need to indicate the bases that were used to create the matrix A, we'll write $A = [T]_\beta^\gamma$.

What we've actually done here was that we used the isomorphisms $\phi_\beta : V \to \mathbf{R}^n$ and $\phi_\gamma : V \to \mathbf{R}^m$ to transfer the linear transformation $T : V \to W$ to a linear transformation $T' : \mathbf{R}^n \to \mathbf{R}^m$. Define T' as $\phi_\gamma \circ T \circ \phi_\beta^{-1}$.

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\phi_\beta"'] & W \arrow[d, "\phi_\gamma"] \\ \mathbf{R}^n \arrow[r, "T'"] & \mathbf{R}^m \end{tikzcd}$$

Whereas T describes the linear transformation $V \to W$ without mentioning the bases or coordinates, T' describes the linear transformation in terms of coordinates. We used those coordinates to describe the entries of the matrix $A = [T]_\beta^\gamma$.

The vector space of linear transformations $\mathrm{Hom}_F(V, W)$. Let's use the notation $\mathrm{Hom}_F(V, W)$ for the set of linear transformations $V \to W$, or more simply $\mathrm{Hom}(V, W)$ when there's only one field under consideration.

This is actually a vector space itself. If you have two transformations, $S : V \to W$ and $T : V \to W$, then define their sum by

$$(S + T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v}).$$

Check that this sum is a linear transformation by verifying that

(1) for two vectors $\mathbf{u}$ and $\mathbf{v}$ in V,

$$(S + T)(\mathbf{u} + \mathbf{v}) = S(\mathbf{u} + \mathbf{v}) + S(\mathbf{u} + \mathbf{v}), \text{ and}$$

(2) for a vector $\mathbf{v}$ and a scalar c,

$$(S + T)(c\mathbf{v}) = c(S + T)(\mathbf{v}).$$

Next, define scalar multiplication on $\mathrm{Hom}(V, W)$ by

$$(cS)(\mathbf{v}) = c(S)(\mathbf{v})$$

and show that's a linear transformation.

There are still eight axioms for a vector space that need to be checked to show that with these two operations $\mathrm{Hom}(V, W)$ is, indeed, a vector space. But note how the operations of addition and scalar multiplication were both defined by those operations on W. Since the axioms hold in W, they'll also hold in $\mathrm{Hom}(V, W)$.

The vector space of matrices $M_{m\times n}(F)$. $M_{m\times n}(F)$ is the set of $m \times n$ matrices with entries in the field F. We'll drop the subscript F when the field is understood.

When V and W are endowed with bases β and γ, we have a bijection $\phi_\beta^\gamma : \mathrm{Hom}(V, W) \overset{\simeq}{\to} M_{m\times n}$ where a linear transformation T corresponds to the matrix $A = [T]_\beta^\gamma$. The entries A_{ij} of A describe how to express T evaluated at the β-basis vector $\mathbf{b}_j$ as a linear combination of the γ-basis vectors.

$$(\mathbf{b}_j) = \sum A_{ij}\mathbf{c}_i$$

We'll use this bijection ϕ_β^γ to define addition and scalar multiplication on $M_{m\times n}$, and when we do that, ϕ_β^γ will be an isomorphism of vector spaces.

First consider addition. Given $T : V \to W$ and $S : V \to W$. Let $A = [T]_\beta^\gamma$ be the standard matrix for T, and let $B = [S]_\beta^\gamma$ be the standard matrix for S. Then

$$\begin{aligned} T(\mathbf{b}_j) &= \sum_i A_{ij}\mathbf{c}_i, \text{ and} \\ S(\mathbf{b}_j) &= \sum_i B_{ij}\mathbf{c}_i. \text{ Therefore} \\ (T + S)(\mathbf{b}_j) &= \sum_i (A_{ij} + B_{ij})\mathbf{c}_i. \end{aligned}$$

Thus, the entries in the standard matrix for $T + S$ are sums of the corresponding entries of the standard matrices for T and S. We now know how we should define addition of matrices.

Definition 1. We define addition of two $m \times n$ matrices coordinatewise: $(A + B)_{ij} = A_{ij} + B_{ij}$.

Next, consider scalar multiplication. Given a scalar c and a linear transformation $T : V \to W$, let A be the standard matrix for T. Then

$$\begin{aligned} T(\mathbf{b}_j) &= \sum_i A_{ij}\mathbf{c}_i. \text{ Therefore} \\ (cT)(\mathbf{b}_j) &= \sum_i cA_{ij}\mathbf{c}_i. \end{aligned}$$

Thus, the entries in the standard matrix for cT are each c times the corresponding entries of the standard matrice for T. We now know how we should define scalar multiplication of matrices.

Definition 2. We define scalar multiplication of a scalar c and an $m \times n$ matrix A coordinatewise: $(cA)_{ij} = cA_{ij}$.

With these definitions of addition and scalar multiplication, $M_{m\times n}(F)$ becomes a vector space over F, and it is isomorphic to $\mathrm{Hom}_F(V, W)$.

Invariants. Note that the isomorphism

$$\mathrm{Hom}(V, W) \cong M_{m \times n}$$

depends on the ordered bases you choose. With different β and γ you'll get different matrices for the same transformation.

In some sense, the transformation holds intrinsic information, while the matrix is a description which varies depending on the bases you happen to choose.

When you're looking for properties of the transformation, they shouldn't be artifacts of the chosen bases but properties that hold for all bases.

When we get to looking for properties of transformations, we'll make sure that they're invariant under change of basis.

Math 130 Home Page at
`http://math.clarku.edu/~ma130/`