The definition of cross products. The cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is an operation that takes two vectors $\mathbf{u}$ and $\mathbf{v}$ in space and determines another vector $\mathbf{u} \times \mathbf{v}$ in space. (Cross products are sometimes called outer products, sometimes called vector products.) Although we’ll define $\mathbf{u} \times \mathbf{v}$ algebraically, its geometric meaning is more understandable.

The cross product $\mathbf{u} \times \mathbf{v}$ is determined by its length and its direction. It’s length is equal to the area of the parallelogram whose sides are $\mathbf{u}$ and $\mathbf{v}$, and that area is the length of $\mathbf{u}$ times the length of $\mathbf{v}$ times the sine of the angle $\theta$ between them. Thus

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

The direction of $\mathbf{u} \times \mathbf{v}$ will be orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$ in a direction determined by a right-hand rule (when the coordinate system is right-handed).

The easiest way to define cross products is to use the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for $\mathbb{R}^3$. If

$$\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k},$$

and

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

then $\mathbf{u} \times \mathbf{v}$ is defined as

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$
which is much easier to remember when you write it as a determinant

\[
\begin{vmatrix}
  u_2 & u_3 \\
v_2 & v_3
\end{vmatrix} 
- \begin{vmatrix}
  u_1 & u_3 \\
v_1 & v_3
\end{vmatrix} \mathbf{i} + \begin{vmatrix}
  u_1 & u_2 \\
v_1 & v_2
\end{vmatrix} \mathbf{k}
\]

\[
\begin{vmatrix}
  i & j & k \\
u_1 & u_2 & u_3 \\
v_1 & v_2 & v_3
\end{vmatrix}
\]

**Properties of cross products.** There are a whole lot of properties that follow from this definition. First of all, it’s anticommutative

\[
\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}),
\]

so any vector cross itself is 0

\[
\mathbf{v} \times \mathbf{v} = 0.
\]

It’s bilinear, that is, linear in each argument, so it distributes over addition and subtraction, 0 acts as zero should, and you can pass scalars in and out of arguments

\[
\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})
\]

\[
(\mathbf{u} \pm \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \pm (\mathbf{v} \times \mathbf{w})
\]

\[
0 \times \mathbf{v} = 0 = \mathbf{v} \times 0
\]

\[
c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})
\]

A couple more properties you can check from the definition, or from the properties already found are that \langle \mathbf{u} \times \mathbf{v} | \mathbf{u} \rangle = 0 and \langle \mathbf{u} \times \mathbf{v} | \mathbf{v} \rangle = 0. Those imply that the vector \mathbf{u} \times \mathbf{v} is orthogonal to both vectors \mathbf{u} and \mathbf{v}, and so it is orthogonal to the plane of \mathbf{u} and \mathbf{v}.

**Standard unit vectors and cross products.** Interesting things happen when we look specifically at the cross products of standard unit vectors. Of course

\[
i \times i = j \times j = k \times k = 0,
\]

since any vector cross itself is 0. But

\[
i \times j = k, \quad j \times k = i, \quad k \times i = j,
\]

and

\[
j \times i = -k, \quad k \times j = -i, \quad i \times k = -j,
\]

all of which follows directly from the definition.
Length of the cross product, areas of triangles and parallelograms. A direct com-
putation (which we’ll omit) shows that
\[ \| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta \]
where \( \theta \) is the angle between the vectors \( \mathbf{u} \) and \( \mathbf{v} \).

Consider a triangle in 3-space where two of the sides are \( \mathbf{u} \) and \( \mathbf{v} \).

![Triangle Diagram]

Taking \( \mathbf{u} \) to be the base of the triangle, then the height of the triangle is \( \| \mathbf{v} \| \sin \theta \), where \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \). Therefore, the area of this triangle is
\[ \text{Area} = \frac{1}{2} \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta = \frac{1}{2} \| \mathbf{u} \times \mathbf{v} \|. \]
(In general, the area of a any triangle is half the product of two adjacent sides and the sine
of the angle between them.)

Area of a parallelogram in \( \mathbf{R}^3 \). Now consider a parallelogram in 3-space where two of
the sides are \( \mathbf{u} \) and \( \mathbf{v} \).

![Parallelogram Diagram]

Of course, if the triangle is doubled to a parallelogram, then the area of the parallelogram is
\( \| \mathbf{u} \times \mathbf{v} \|. \)

Thus, the norm of a cross product is the area of the parallelgram bounded by the vectors.

We now have a geometric characterization of the cross product. The cross product \( \mathbf{u} \times \mathbf{v} \)
is the vector orthogonal to the plane of \( \mathbf{u} \) and \( \mathbf{v} \) pointing away from it in a the direction
determined by a right-hand rule, and its length equals the area of the parallelgram whose
sides are \( \mathbf{u} \) and \( \mathbf{v} \).

Note that \( \mathbf{u} \times \mathbf{v} \) is \( \mathbf{0} \) if and only if \( \mathbf{u} \) and \( \mathbf{v} \) lie in a line, that is, they point in the same
direction or the directly opposite directions.

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