Determinants
Math 130 Linear Algebra
D Joyce, Fall 2015

Introduction to determinants. Every square matrix A has a determinant, denoted either $\det(A)$ or more commonly $|A|$, which is a number that tells a lot about it. We'll see, for instance, that A is an invertible matrix if and only if $|A| \neq 0$. Also, the determinant tells what the transformation described by A does to area. Specifically, the absolute value of the determinant tells you by what factor any region is enlarged.

Determinants of small matrices. Before looking at the general definition for $n \times n$ square matrices, we'll look at the cases when n is small, namely, 2 or 3.

Let A be the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then we define the determinant of A to be

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ &= ad - bc. \end{aligned}$$

So, the determinant of a 2×2 matrix is the product of the two elements on the major diagonal minus the product of the two elements on the minor diagonal. We've already seen an application of determinants before in the computation of the inverse of a 2×2 matrix.

Now let A be a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Then we define the determinant of A to be $\det(A) =$ $|A| =$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi.$$

There are six terms in this determinant. Each term is a product of three elements, one element chosen out of each row and column. All six possible ways of choosing one element out of each row and column are included. Three have minus signs and three have plus signs.

Historical note. It's very interesting that determinants *predate* matrices. The study of determinants was going well before anyone had any use for matrices. Indeed, determinants were one of the primary reasons why the theory of matrices was created.

Matlab can find determinants. Let's see what it gets for one using the `det` function.

```
>> A=[1,3,5,7;2,4,8,6;0,1,5,3;1,1,0,0]

A =
      1      3      5      7
      2      4      8      6
      0      1      5      3
      1      1      0      0

>> det(A)

ans =
    -2
```

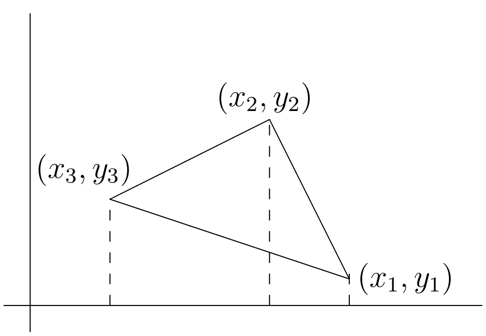
An interesting matrix is the Hilbert matrix generated in Matlab using the `hilb` function. Note that Matlab displays numbers decimally by default, but you can change that with the instruction `format rat`.

```
>> H = hilb(5)

H =
```

```
 1.0000 0.5000 0.3333 0.2500 0.2000
 0.5000 0.3333 0.2500 0.2000 0.1667
 0.3333 0.2500 0.2000 0.1667 0.1429
 0.2500 0.2000 0.1667 0.1429 0.1250
 0.2000 0.1667 0.1429 0.1250 0.1111

>> format rat
>> H

H =
 1     1/2   1/3   1/4   1/5
 1/2   1/3   1/4   1/5   1/6
 1/3   1/4   1/5   1/6   1/7
 1/4   1/5   1/6   1/7   1/8
 1/5   1/6   1/7   1/8   1/9

>> det(H)

ans =
       1/266716800000
```

That's one small determinant!

Areas of triangles and parallelograms in terms of determinants. You can determine the area of a triangle in the plane with vertices at $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3(x_3, y_3)$.



If the vertices are at the position shown, where they go around the triangle in a counterclockwise direction, you can determine the area by adding the areas of two trapezoids with tops P_1P_2 and P_2P_3, then subtracting the area of the trapezoid with top P_3P_1.

The trapezoid with top P_1P_2 has area

$$\tfrac{1}{2}(y_1 + y_2)(x_1 - x_2),$$

that with top P_2P_3 has area

$$\tfrac{1}{2}(y_2 + y_3)(x_2 - x_3),$$

and that with top P_3P_1 is the negation of

$$\tfrac{1}{2}(y_3 + y_1)(x_3 - x_1).$$

Adding those together gives

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

which equals half the value value of a 3×3 determinant

$$\text{Area of triangle} = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

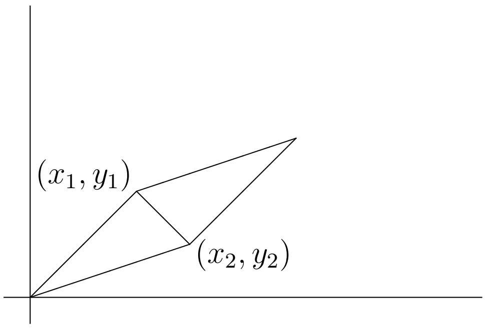
Note that if you exchange the names of two of the vertices, then the value of the determinant is negated. That means that you will either get the area of the triangle or the negation of the area depending on the order that you name the vertices. If you name the vertices as you go around the triangle in a counterclockwise direction, you'll get the area, but if you name them in a clockwise direction, it's the negation of the area. Right now, we're not interested in the sign of the determinant, so just take the absolute value to get the actual area, but that sign becomes very important when you want to know the orientation of the triangle. When the sign is included with the area, then this is called the *signed area* of the triangle.

Likewise, the area of a parallelogram where any three of its vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is the absolute value of the determinant without taking half of it.

Note that if the vertex (x_3, y_3) is placed at the origin $(0, 0)$, then the determinant simplifies to

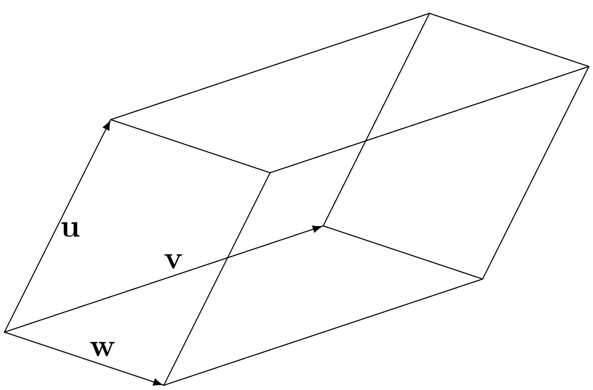
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}.$$

That gives us a geometric interpretation of the 2×2 determinant as the signed area a certain parallelogram with two sides being the vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ which are the columns of the matrix.



Geometric interpretation of determinants in higher dimensions. The last statement for areas of parallelograms can be extended to higher dimensions. You can find the volume of a parallelepiped in $\mathbf{R}^3$ as the determinant of a 3×3.

Consider a parallelepiped whose edges are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.



The signed volume of this parallelepiped is

$$\text{Volume} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}.$$

We won't prove this now, but we may prove it later in the course.

Likewise, the n-dimensional analogue of signed volume for an n-dimensional parallelepiped whose edges are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the determinant of the matrix whose columns are those n vectors.

An application to multivariable calculus: Jacobians. Jacobians are determinants used in substitution when you have multiple integrals. You'll see them when you study multivariable calculus (or you already have if you've already studied it).

Recall that in calculus of one variable, when you use substitution, you insert a derivative. When you substitute $u = g(x)$, with $du = \dfrac{du}{dx}\,dx$, the substitution formula looks like

$$\int f(u)\,du = \int f(g(x))\,\frac{du}{dx}\,dx.$$

In higher dimensions, that derivative is replaced by a determinant of partial derivatives. Here's what the change of variables formula looks like in dimension 2.

$$\begin{aligned} &\iint f(u,v)\,du\,dv \\ = &\iint f(g(x,y),h(x,y)) \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} \,dx\,dy \end{aligned}$$

where $u = g(x,y)$, $v = h(x,y)$. Here $\frac{\partial u}{\partial x}$ is something called the partial derivative of u with respect to x, that is, the derivative of $g(x,y)$ with respect to x, and $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$, and $\frac{\partial v}{\partial y}$ are three other partial derivatives. We can't go into much detail about this in this course, but to more fully explain what's going on, we'll look at one example.

The reason that particular determinant appears in the integral is that it represents the signed area of a parallelogram with the two edges $\mathbf{u}' = \left(\frac{du}{dx}, \frac{du}{dy}\right)$ and $\mathbf{v}' = \left(\frac{dv}{dx}, \frac{dv}{dy}\right)$.

Example 1. We'll look at how to make a change of variables in this double integral

$$\iint \sin(u+v)\cos(u-v)\,du\,dv$$

by making the substitution $u = (x+y)/2$ and $v = (x-y)/2$ so that $x = u+v$ and $y = u-v$.

First compute the four partial derivatives. The first one, $\frac{\partial u}{\partial x}$ is the partial derivative of $(x+y)/2$ with respect to x, which is $\frac{1}{2}$, because when you take partial derivatives with respect to x, you treat y as a constant. The second is $\frac{\partial v}{\partial x}$, the partial derivative of $(x-y)/2$ with respect to x, and that's $\frac{1}{2}$, too. The third is $\frac{\partial u}{\partial y}$, the partial derivative of $(x+y)/2$ with respect to y, and that's $\frac{1}{2}$ again. The fourth is $\frac{\partial v}{\partial y}$, the partial derivative of $(x-y)/2$ with respect to y, which is $-\frac{1}{2}$.

Therefore, the Jacobian is

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

When you use that in the formula for substitution in double integrals, that gives

$$\begin{aligned} & \iint \sin(u+v)\cos(u-v)\,du\,dv \\ = & \iint \sin x \cos y \,(-\tfrac{1}{2})\,dx\,dy \end{aligned}$$

That's enough to indicate one way that determinants will be used in multivariable calculus.

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