Linear Combinations, Basis, Span, and Independence
Math 130 Linear Algebra
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We’re interested is pinning down what it means for a vector space to have a basis, and that’s described in terms of the concept of linear combination. Span and independence are two more related concepts.

Generally, in mathematics, you say that a linear combination of things is a sum of multiples of those things. So, for example, one linear combination of the functions \( f(x), g(x), \) and \( h(x) \) is

\[
2f(x) + 3g(x) - 4h(x).
\]

**Definition 1** (Linear combination). A **linear combination** of vectors \( v_1, v_2, \ldots, v_k \) in a vector space \( V \) is an expression of the form

\[
c_1v_1 + c_2v_2 + \cdots + c_kv_k
\]

where the \( c_i \)'s are scalars, that is, it’s a sum of scalar multiples of them. More generally, if \( S \) is a set of vectors in \( V \), not necessarily finite, then a linear combination of \( S \) refers to a linear combination of some finite subset of \( S \).

Of course, differences are allowed, too, since negations of scalars are scalars.

We can use linear combinations to characterize subspaces as mentioned previously when we talked about subspaces.

**Theorem 2.** A nonempty subset \( W \) of a vector space \( V \) is a subspace of \( V \) if and only if \( W \) is closed under linear combinations, that is, whenever \( w_1, w_2, \ldots, w_k \) all belong to \( W \), then so does each linear combination \( c_1w_1 + c_2w_2 + \cdots + c_kw_k \) of them belong to \( W \).

**A basis for a vector space.** You know some bases for vector spaces already even if you haven’t know them by that name.

For instance, in \( \mathbb{R}^3 \) the three vectors \( \mathbf{i} = (1,0,0) \) which points along the \( x \)-axis, \( \mathbf{j} = (0,1,0) \) which points along the \( y \)-axis, and \( \mathbf{k} = (0,0,1) \) which points along the \( z \)-axis together form the **standard basis** for \( \mathbb{R}^3 \). Every vector \((x,y,z)\) in \( \mathbb{R}^3 \) is a unique linear combination of the standard basis vectors

\[
(x, y, z) = xi + yj + zk.
\]

That’s the one and only linear combination of \( i, j \), and \( k \) that gives \((x,y,z)\). (Why?) We’ll generally use Greek letters like \( \beta \) and \( \gamma \) to distinguish bases (‘bases’ is the plural of ‘basis’) from other subsets of a set. Thus \( \epsilon = \{i,j,k\} \) is the standard basis for \( \mathbb{R}^3 \). We’ll want our bases to have an ordering to correspond to a coordinate system. So, for this basis \( \epsilon \) of \( \mathbb{R}^3 \), \( i \) comes before \( j \), and \( j \) comes before \( k \).

The plane \( \mathbb{R}^2 \) has a standard basis of two vectors, namely, \( \epsilon = \{i,j\} \) where \( i = (1,0) \) and \( j = (0,1) \). (Although we’re using \( i \) and \( j \) for different things, you can tell what’s meant by context.)

There is an analogue for \( \mathbb{R}^n \). Its standard basis is

\[
\epsilon = \{e_1, e_2, \ldots, e_n\}
\]

where

\[
e_1 = (1,0,\ldots,0),
\]

\[
e_2 = (0,1,\ldots,0),
\]

\[
\cdots
\]

\[
e_n = (0,0,\ldots,1).
\]

Sometimes it’s nice to have a notation without the ellipsis \( (\ldots) \), and the Kronecker delta symbol helps here. Let \( \delta_{ij} \) be defined by

\[
\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

Then the \( j^{\text{th}} \) coordinate \( e_{ij} \) of the \( i^{\text{th}} \) standard unit vector \( e_i \) is \( \delta_{ij} \).

Coordinates are related to bases. Let \( \mathbf{v} \) be a vector in \( \mathbb{R}^n \). It can be uniquely written as a linear
combinations of the standard basis vectors
\[ \mathbf{v} = v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2 + \cdots + v_n \mathbf{e}_n \]
and the coefficients that appear in this unique linear combination are the coordinates of \( \mathbf{v} \)
\[ \mathbf{v} = (v_1, v_2, \ldots, v_n). \]

That leads us to the definition of for the concept of \textit{basis} of a vector space. Whenever we used a basis in conjunction with coordinates, we’ll need an ordering on it, but for other purposes the ordering won’t matter.

\textbf{Definition 3.} An \textit{(ordered) subset}
\[ \beta = \{ \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n \} \]
of a vector space \( V \) is an \textit{(ordered) basis} of \( V \) if each vector \( \mathbf{v} \) in \( V \) may be uniquely represented as a linear combination of vectors from \( \beta \)
\[ \mathbf{v} = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + \cdots + v_n \mathbf{b}_n. \]

For an ordered basis, the coefficients in that linear combination are called the \textit{coordinates} of the vector with respect to \( \beta \).

Later on, when we study coordinates in more detail, we’ll write the coordinates of a vector \( \mathbf{v} \) as a column vector and give it a special notation
\[ [\mathbf{v}]_\beta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \]

Although we have a standard basis for \( \mathbb{R}^n \), there are other bases.

\textbf{Example 4.} For example, the two vectors \( \mathbf{b}_1 = (1, 1) \) and \( \mathbf{b}_2 = (1, -1) \) form a basis \( \beta = (\mathbf{b}_1, \mathbf{b}_2) \) for \( \mathbb{R}^2 \). Each vector \( \mathbf{v} = (v_1, v_2) \) can be written as a unique linear combination of them, namely
\[ \mathbf{v} = (v_1, v_2) = \frac{1}{2}(v_1 + v_2)\mathbf{b}_1 + \frac{1}{2}(v_1 - v_2)\mathbf{b}_2. \]

So the \( \beta \)-coordinates of \( \mathbf{v} \) are
\[ [\mathbf{v}]_\beta = \begin{bmatrix} v_1 + v_2 \\ v_1 - v_2 \end{bmatrix} \]

So, for instance, the vector which has standard coordinates \((2, 4)\) has the \( \beta \)-coordinates \([3, -1]\) because \((2, 4) = 3\mathbf{b}_1 - \mathbf{b}_2\).

There are lots of other bases for \( \mathbb{R}^2 \). In fact, if you take any two vectors \( \mathbf{b}_1 \) and \( \mathbf{b}_2 \) that don’t lie on a line, they’ll form a basis.