Linear Differential Equations
Math 130 Linear Algebra
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It's important to know some of the applications of linear algebra, and one of those applications is to homogeneous linear differential equations with constant coefficients.

What is a differential equation? It's an equation where the unknown is a function and the equation is a statement about how the derivatives of the function are related.

Perhaps the most important differential equation is the exponential differential equation. That's the one that says a quantity is proportional to its rate of change. If we let t be the independent variable, $f(t)$ the quantity that depends on t, and k the constant of proportionality, then the differential equation is

$$f'(t) = kf(t).$$

Its general solution is

$$f(t) = Ae^{kt}$$

where A is an arbitrary constant. This differential equation has applications as the exponential model of population growth, radioactive decay, compound interest, Newton's law of cooling, and lot's more.

Another important differential equation is the one that the sine and cosine function satisfies. For both of those functions, their second derivative is their negation. They satisfy the differential equation

$$f''(t) = -f(t).$$

That's called a *second order* differential equation since the second derivative is involved. The general solution to this differential equation is

$$f(t) = A\cos t + B\sin t$$

where A and B are arbitrary constants.

Both of these equations are *homogeneous linear differential equations with constant coefficients.* An n^{th}-order linear differential equation is of the form

$$a_n f^n(t) + \cdots + a_2 f''(t) + a_1 f'(t) + a_0 f(t) = b.$$

that is, some linear combination of the derivatives up through the n^{th} derivative is equal to b. In general, the coefficients $a_n, \ldots, a_1, a_0$, and b can be any functions of t, but when they're all scalars (either real or complex numbers), then it's a linear differential equation with constant coefficients. When b is 0, then it's homogeneous.

For the rest of this discussion, we'll only consider at homogeneous linear differential equations with constant coefficients. The two example equations are such; the first being $f' - kf = 0$, and the second being $f'' + f = 0$.

What do they have to do with linear algebra? Their solutions form vector spaces, and the dimension of the vector space is the order n of the equation.

The first example was the first-order equation $f' - kf = 0$. Its solutions were Ae^{kt} where A was an arbitrary constant. The set of these solutions is a vector space because the 0 function is one solution, they're closed under addition, and they're closed under multiplication by constants.

The second example was the second-order equation $f'' + f = 0$. Its solutions, $A\cos t + B\sin t$ form a two-dimensional vector space.

In general the set of solutions will always be a vector space. The zero function is always a solution; that's because we're only considering homogeneous differential equations. If f and g both satisfy the equation, then so does $f + g$. Finally, if c is a scalar and f satisfies the equation, so does cf.

So, how do you solve these equations, and why does the dimension of the solution space equal the order of the equation? Let's take an example to illustrate it. It should be general enough to show the method, but not two complicated. Consider the fourth-order equation

$$f'''' - 4f''' + 5f'' - 4f' - 4f = 0.$$

The trick is to treat derivatives as differential operators—things which operate on functions. We could denote this operator as $\dfrac{d}{dt}$ as is common in calculus, but a simple uppercase D makes it easier to see what's going on. Then the differential equation looks like

$$D^4 f - 4D^3 f + 5D^2 f - 4Df - 4f = 0,$$

and that can be rewritten as

$$(D^4 - 4D^3 + 5D^2 - 4D - 4)f = 0.$$

Now, I chose this example to factor nicely. The differential operator $D^4 - 4D^3 + 5D^2 - 4D - 4$ factors as $(D^2 + 1)(D^2 - 4D + 4)$, which further factors as $(D^2 + 1)(D - 2)^2$. That first quadratic polynomial, $D^2 + 1$, is irreducible over $\mathbf{R}$ but factors over $\mathbf{C}$, so to make things easier, we'll work over $\mathbf{C}$. That gives us the factorization into linear factors

$$(D - i)(D + i)(D - 2)^2.$$

The Fundamental Theorem of Algebra assures us that we can always factor an n^{th} degree polynomial into linear factors over $\mathbf{C}$, so what happened in this example will also happen in the general case. The Fundamental Theorem of Algebra connnects the order of the differential equation to the dimension of the solution space.

Now we can break down the differential equations into linear pieces. We can solve $(D - i)f = 0$ since it's just the exponential differential equation. It has solutions $f(t) = Ae^{it}$. Likewise $(D+i)f = 0$ has the solutions $f(t) = Be^{-it}$. That gives us solutions for $(D^2 + 1)f = 0$, namely $Ae^{it} + Be^{-it}$. Note that we already saw a different description of the solutions for this equation, $f'' - f = 0$, and that was as $A \cos t + B \sin t$. The connection between the two forms of the solutions is Euler's identity

$$e^{it} = \cos t + i \sin t.$$

You can use Euler's identity to convert between the two forms.

We still have the other factor $(D - 2)^2$ to deal with. The double root at 2 complicates the solutions. I'll leave out the derivation of the solution to the differential equation $(D - 2)^2 t = 0$, and just give its general solution, $f(t) = Cte^{2t} + De^{t2}$. You can check that it works. Analogous solutions can be found when there are multiple roots rather than just double roots.

In summary, this example differential equation, $f'''' - 4f''' + 5f'' - 4f' - 4f = 0$, has the general solution

$$f(t) = Ae^{it} + Be^{-it} + Cte^{2t} + De^{t2}.$$

We haven't shown that there aren't any other solutions, but we'll skip that part since we're just doing a survey on the topic, not a whole course.

Dynamical systems. You might ask, are these really important? Yes. Linear dynamical systems are modelled using these equations.

In such a system you have several things that change over time. Suppost they're the quantities x, y, and z. Some of these things affect others positively, some negatively. If x affects y positively, then y' will have a term kx where k is positive, but if negatively, then k is negative. A linear dynamical system is a system of linear equations like

$$\left\{ \begin{array}{rclcrcr} x' & = & & + & 2y & - & z \\ y' & = & 2x & & & & \\ z' & = & & & y & - & 3z \end{array} \right.$$

For this example, x and y affect each other positively in a positive feedback loop, and y positively affects z, but z negatively affects x and itself.

Although it's a system of three linear differential equations, it's equivalent to one third order differential equation.

Dynamical systems like this are used in physics, chemistry, biology, economics, and almost any subject that calls itself a science.

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