

Some linear transformations on \mathbf{R}^2
 Math 130 Linear Algebra
 D Joyce, Fall 2015

Let's look at some some linear transformations on the plane \mathbf{R}^2 . We'll look at several kinds of operators on \mathbf{R}^2 including reflections, rotations, scalings, and others.

We'll illustrate these transformations by applying them to the leaf shown in figure 1. In each of the figures the x -axis is the red line and the y -axis is the blue line.

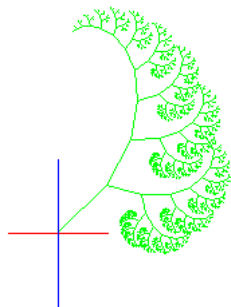


Figure 1: Basic leaf

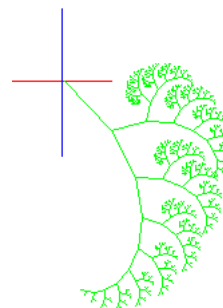


Figure 2: Reflected across x -axis

Example 1 (A reflection). Consider the 2×2 matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Take a generic point $\mathbf{x} = (x, y)$ in the plane, and write it as the column vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Then the matrix product $A\mathbf{x}$ is

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$

Thus, the matrix A transforms the point (x, y) to the point $T(x, y) = (x, -y)$. You'll recognize this right away as a reflection across the x -axis.

The x -axis is special to this operator as it's the set of fixed points. In other words, it's the 1-eigenspace. The y -axis is also special as every point $(0, y)$ is sent to its negation $-(0, y)$. That means the y -axis is an eigenspace with eigenvalue -1 , that is, it's the -1 -eigenspace. There aren't any other eigenspaces. A reflection always has these two eigenspaces a 1-eigenspace and a -1 -eigenspace.

Every 2×2 matrix describes some kind of geometric transformation of the plane. But since the origin $(0, 0)$ is always sent to itself, not every geometric transformation can be described by a matrix in this way.

Example 2 (A rotation). The matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ determines the transformation that sends the vector $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ to the vector $\mathbf{x} = \begin{bmatrix} -y \\ x \end{bmatrix}$. It determines the linear operator $T(x, y) = (-y, x)$. In particular, the two basis vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are sent to the vectors $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{e}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ respectively. Note that these are the first and second columns of A . You'll recognize this transformation as a rotation around the origin by 90° . (As always, the convention is to always take counterclockwise rotations to be by a positive number of degrees, but clockwise ones by a negative number of degrees.)

There are no eigenvalues for this rotation. There are no nontrivial eigenvectors at all. (We don't consider $\mathbf{0}$ since it's always sent to itself.) If, however, you use complex scalars, there is one. The vector $(1, -i)$ is sent to $(i, 1)$, so it's an i -eigenvector, and $(1, i)$ is sent to $(-i, 1)$, so it's a $-i$ -eigenvector. We'll look at complex eigenvalues later when we study linear operators in detail.

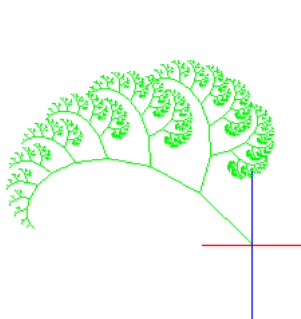


Figure 3: Rotated 90°

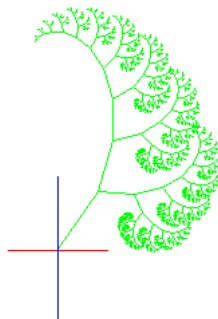


Figure 4: Rotated 10°

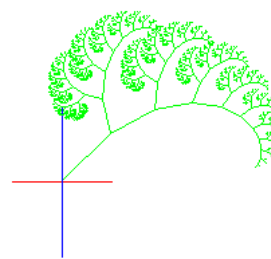


Figure 5: Reflected across the line $y = x$

Example 3 (Other rotations). Rotations by other angles θ can be described with the help of trig functions. The matrix $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ describes a rotation of the plane by an angle of θ .

For example, the matrix that describes a rotation of the plane around the origin of 10° counterclockwise is $\begin{bmatrix} \cos 10^\circ & -\sin 10^\circ \\ \sin 10^\circ & \cos 10^\circ \end{bmatrix} = \begin{bmatrix} 0.9848 & -0.1736 \\ 0.1736 & 0.9848 \end{bmatrix}$ since $\sin 10^\circ = 0.1736$ and $\cos 10^\circ = 0.9848$.

Note that for any rotation, any circle centered at the origin is invariant. A set is said to be *invariant* under a transformation if it is sent to itself. The individual points in the set may be moved, but they must stay within the set.

All rotations and reflections preserve distance. That means that the distance between any two vectors \mathbf{v} and \mathbf{w} is the same as the distance between their images $A\mathbf{v}$ and $A\mathbf{w}$. Such transformations are called *rigid transformations* or *isometries*.

Example 4 (Other reflections). We've already seen that the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ describes a reflection across the x -axis. Likewise, $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ describes a reflection across the y -axis. There's a 2×2 matrix for reflection across any line through the origin. What matrix describes a reflection across the line $y = x$? What are the eigenspaces for this reflection?

Example 5 (Expansions). Not all linear operators preserve distance. For instance, contractions and expansions don't. The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ sends a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to the vector $\begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Thus, every point is sent twice as far away from the origin. That's an expansion by a factor of 2. Note that every vector is a 2-eigenvector. In other words, all of \mathbf{R}^2 is the 2-eigenspace.

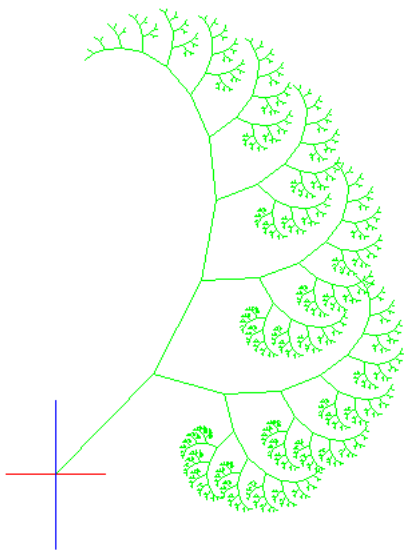


Figure 6: Expansion



Figure 7: Contraction

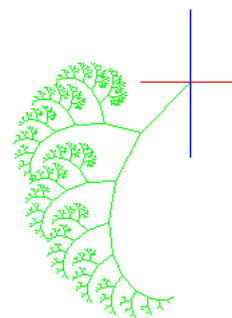


Figure 8: Half turn

A *scalar matrix* is a multiple of the identity matrix like this one was. If λ is a scalar, then

$$\lambda I = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

is a scalar matrix with scalar λ .

Every scalar matrix where the scalar is greater than 1 describes an expansion.

Example 6 (Contractions). When the scalar is between 0 and 1, then the matrix describes a contraction. For instance $\begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}$ moves vectors toward the origin half as far away as where they started.

Example 7 (Point inversion a.k.a a half turn). The particular scalar matrix $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ sends a point to the other side of the origin, but the same distance away from the origin. That's the same as a 180° rotation. This operation is called a *half turn* or *point inversion*.

Other scalar matrices with negative scalars describe transformations that can be thought of as compositions of point inversions and either expansions or contractions.

Sometimes the terms “dilation” and “dilatation” is used for any of these transformations determined by scalar matrices.

Example 8 (Projections). Consider the 2×2 matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It sends the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to the vector $\begin{bmatrix} x \\ 0 \end{bmatrix}$. This particular transformation is a projection from 2-space onto the x -axis that forgets the y -coordinate. Projections have 1-eigenspaces and 0-eigenspaces. What are they for this projection?

This projection will map our leaf to the x -axis as shown in figure 9.

Example 9 (Shear transformations). The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ describes a “shear transformation” that fixes the x -axis, moves points in the upper half-plane to the right, but moves points in the lower half-plane to the left.

In general, a shear transformation has a line of fixed points, its 1-eigenspace, but no other eigenspace. Shears are deficient in that way.

Note that any line parallel to the line of fixed points is invariant under the shear transformation.

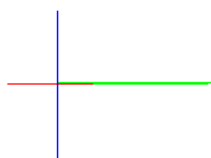


Figure 9: Projection to x -axis

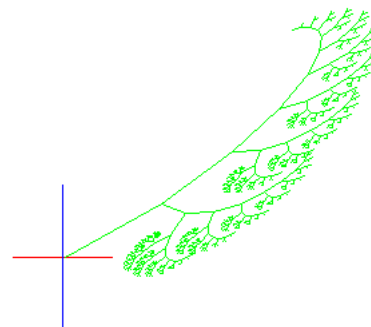


Figure 10: A shear transformation

Example 10 (Stretch and squeeze). Another interesting transformation is described by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$ which sends the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ to the vector $\begin{bmatrix} 2x \\ 0.5y \end{bmatrix}$. The plane is transformed by stretching horizontally by a factor of 2 at the same time as it's squeezed vertically. (What are its eigenvectors and eigenvalues?)

For this transformation, each hyperbola $xy = c$ is invariant, where c is any constant.

These last two examples are plane transformations that preserve areas of figures, but don't preserve distance.

If you randomly choose a 2×2 matrix, it probably describes a linear transformation that doesn't preserve distance and doesn't preserve area. Let's look at one more example before leaving dimension 2.

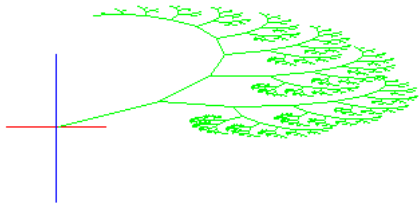


Figure 11: Stretch and squeeze

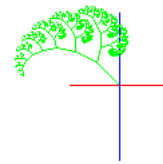


Figure 12: A rotary contraction

Example 11 (Rotary contraction). If you compose a rotation and a contraction, you get a rotary contraction. The matrix $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ describes a rotation about the origin by an angle of θ , while the matrix $\begin{bmatrix} .5 & 0 \\ 0 & 0.5 \end{bmatrix}$ contracts the plane toward the origin by $\frac{1}{2}$, so the product of the matrices $\begin{bmatrix} 0.5 \cos \theta & -0.5 \sin \theta \\ 0.5 \sin \theta & 0.5 \cos \theta \end{bmatrix}$ describes the composition of contracting and rotating. (In this case, the order of composing them doesn't matter since the two operations commute, but usually order makes a difference.) The result is a kind of spiral transformation. If you take any point \mathbf{x} and repeatedly apply this transformation, the orbit that is produced is a spiral toward the origin.

Logarithmic spirals whose equations in polar coordinates $r = c2^{-2\theta/\pi}$ are invariant subsets of this rotary contraction, where c is any constant.

Transformations of \mathbf{R}^3 . A 3×3 matrix describes a transformation of space, that is, a 3-D operator. There are many kinds of such transformations, some isometries, some not.

Isometries include (1) reflections across planes that pass through the origin, (2) rotations around lines that pass through the origin, and (3) rotary reflections. Rotary reflections are compositions of a reflection across a plane and rotation around an axis perpendicular to that plane.

Most of the linear transformations on \mathbf{R}^3 aren't isometries. They include projections, expansions and contractions, shears, rotary expansions and contractions, and many others.

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