Sets
Math 130 Linear Algebra
D Joyce, Fall 2015

Just a little bit about sets. We'll use the language of sets throughout the course, but we're not using much of set theory. Still, it would be useful to know a little bit about it.

A set itself is just supposed to be something that has elements. It doesn't have to have any structure but just have elements. The elements can be anything, but usually they'll be things of the same kind.

If you've only got one set, however, there's no need to even mention sets. It's when several sets are under consideration that the language of sets becomes useful.

There are ways to construct new sets, too, and these constructions are important. The most important of these is a way to collect some of the elements in a set to form another set, a subset of the first.

Examples. Let's start with sets of numbers. There are ways of constructing these sets, but let's not deal with that now. Let's assume that we already have these sets.

The natural numbers. These are the counting numbers, that is, whole positive numbers. 1 is the first natural number, 2 the second, 3, the third, etc. We'll use $\mathbf{N}$ to denote the set of all natural numbers. Some people like to include 0 in the natural numbers, but I follow Dedekind who started with 1. There is a structure on $\mathbf{N}$, namely there are operations of addition, subtraction, etc., but as a set, it's just the numbers. You'll often see $\mathbf{N}$ defined as

$$\mathbf{N} = \{1, 2, 3, \ldots\}$$

which is read as $\mathbf{N}$ is the set whose elements are 1, 2, 3, and so forth. That's just an informal way of describing what $\mathbf{N}$ is. A complete description couldn't get away with "and so forth." If you want to see all of what "and so forth" entails, you can read Dedekind's 1888 paper *Was sind und was sollen die Zahlen?* and my comments on it. In that article he starts off developing set theory and ends up with the natural numbers.

The real numbers. These include all positive numbers, negative numbers, and 0. Besides the natural numbers, their negations and 0 are included, fractions like $\frac{22}{7}$, algebraic numbers like $\sqrt{5}$, and transcendental numbers like π, e. If a number can be named decimally with infinitely many digits, then it's a real number. We'll use $\mathbf{R}$ to denote the set of all real numbers. Like $\mathbf{N}$, $\mathbf{R}$ has lots of operations and functions associated with it, but treated as a set, all it has is its elements, the real numbers.

Note that $\mathbf{N}$ is a subset of $\mathbf{R}$ since every natural number is a real number.

Subsets. If you have a set and a language to talk about elements in that set, then you can form subsets of that set by properties of elements in that language.

For instance, we have arithmetic on $\mathbf{R}$, so solutions to equations are subsets of $\mathbf{R}$. The solutions to the equation $x^3 = x$ are 0, 1, and -1. We can describe its solution set using the notation

$$S = \{x \in \mathbf{R} \,|\, x^3 = x\}$$

which is read as "S is the set of x in $\mathbf{R}$ such that $x^3 = x$." We could also describe that set by listing its elements, $S = \{0, 1, -1\}$. When you name a set by listing its elements, the order that you name them doesn't matter. We could have also written $S = \{-1, 0, 1\}$ for the same set.

Open and closed intervals in $\mathbf{R}$ are also subsets of $\mathbf{R}$. For example,

$$\begin{aligned} (3, 5) &= \{x \in \mathbf{R} \,|\, 3 < x < 5\} \\ [3, 5] &= \{x \in \mathbf{R} \,|\, 3 \le x \le 5\} \end{aligned}$$

There are a couple of notations for subsets. We'll use the notation $A \subseteq S$ to say that A is a subset of S. We allow $S \subseteq S$, that is, we consider a set S to be a subset of itself. If a subset A doesn't include all the elements of S, then A is called a *proper* subset of S. The only subset of S that's not a proper subset is S itself. We'll use the notation $A \subset S$ to indicate that A is a proper subset of S.

(Warning. There's an alternate notational convention for subsets. In that notation $A \subset S$ means A is any subset of S, while $A \subsetneq S$ means A is a proper subset of S. I prefer the the notation we're using because it's analogous to the notations $\leq$ for less than or equal, and $<$ for less than.)

Operations on subsets. Frequently you deal with several subsets of a set, and there are operations of intersection, union, and difference that describe new subsets in terms of previously known subsets.

The intersection $A \cap B$ of two subsets A and B of a given set S is the subset of S that includes all the elements that are in both A and B:

$$A \cap B = \{x \in S \,|\, x \in A \text{ and } x \in B\}.$$

The union $A \cup B$ of two subsets A and B of a given set S is the subset of S that includes all the elements that are in A or in B or in both:

$$A \cup B = \{x \in S \,|\, x \in A \text{ or } x \in B\}.$$

As usual in mathematics, the word "or" means an inclusive or and implicitly includes "or both."

The difference $A - B$ of two subsets A and B of a given set S is the subset of S that includes all the elements that are in A but not in B:

$$A - B = \{x \in S \,|\, x \in A \text{ and } x \notin B\}$$

There's also the complement of a subset A of a set S. The complement is just $S - A$, all the elements of S that aren't in A. When the set S is understood, the complement of A often is denoted more simply as either $\overline{A}$ or A^c.

These operations satisfy lots of identities. I'll just name a couple of important ones.

DeMorgan's laws describe a duality between intersection and union. They can be written as

$$\begin{aligned} \overline{A \cap B} &= \overline{A} \cup \overline{B} \\ \overline{A \cup B} &= \overline{A} \cap \overline{B} \end{aligned}$$

The distributivity laws say that intersection and union each distribute over the other

$$\begin{aligned} (A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C &= (A \cup C) \cap (B \cup C) \end{aligned}$$

Products of sets. So far we've looked at creating sets within set. There are some operations on sets that create bigger sets, the most important being creating products of sets. These depend on the concept of ordered pairs of elements. The notation for ordered pair (a, b) of two elements extends the usual notation we use for coordinates in the xy-plane. The important property of ordered pairs is that two ordered pairs are equal if and only if they have the same first and second coordinates:

$$(a, b) = (c, d) \text{ iff } a = c \text{ and } b = d.$$

The product of two sets S and T consists of all the ordered pairs where the first element comes from S and the second element comes from T:

$$S \times T = \{(a, b) \,|\, a \in S \text{ and } b \in T\}.$$

Thus, the usual xy-plane is $\mathbf{R} \times \mathbf{R}$, usually denoted $\mathbf{R}^2$.

Besides binary products $S \times T$, you can analogously define ternary products $S \times T \times U$ in terms of triples (a, b, c) where $a \in S$, $b \in T$, and $c \in U$, and higher products, too.

Sets of subsets; power sets. Another way to create bigger sets is to form sets of subsets. If you collect all the subsets of a given set S into a set, then the set of all those subsets is called the *power set* of S, denotes $\mathcal{P}(S)$ or sometimes 2^S.

For example, let S be a set with 3 elements, $S = \{a, b, c\}$. Then S has eight subsets. There are three singleton subsets, that is, subsets having exactly one element, namely $\{a\}$, $\{b\}$, and $\{c\}$. There are three subsets having exactly two elements, namely $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. There's one subset having all three elements, namely S itself. And there's one subset that has no elements. You could denote it $\{\}$, but it's always denoted $\emptyset$ and called the *empty set* or *null set*. Thus, the power set of S has eight elements

$$\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, S\}.$$

Functions and function sets. A function $f : S \to T$ from a set S to a set T can be identified with its graph. Its graph is a particular subset of the product set $S \times T$, namely, the subset

$$\{(x, y) \,|\, x \in S \text{ and } y = f(x)\}.$$

You can tell which subsets A of $S \times T$ are graphs of functions. They're the ones with the following property: for each $x \in S$, there is exactly ordered pair in A whose first element is x.

It's convenient to identify functions with such graphs.

All the functions from S to T can be collected together to form a set, sometimes called a function set, and denoted either T^S or $\mathcal{F}(S, T)$. Since each function is a subset of $S \times T$, this function set is actually a subset of the power set of $S \times T$, that is, $\mathcal{F}(S, T) \subseteq \mathcal{P}(S \times T)$.

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