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Properties of Vector Spaces
Math 130 Linear Algebra

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We defined a vector space as a set equipped with the binary operations of addition and scalar multiplication, a constant vector $\mathbf{0}$, and the unary operation of negation, which satisfy several axioms. Here are the axioms again, but in abbreviated form. They hold for all vectors $\mathbf{v}$ and $\mathbf{w}$ and for all scalars c and d.

1. $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$

2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$

3. $\mathbf{v}+\mathbf{0}=\mathbf{v}=\mathbf{0}+\mathbf{v}$

4. $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$

5. $1\mathbf{v}=\mathbf{v}$

6. $c(d\mathbf{v})=(cd)\mathbf{v}$

7. $c(\mathbf{v}+\mathbf{w})=c\mathbf{v}+c\mathbf{w}$

8. $(c+d)\mathbf{v}=c\mathbf{v}+d\mathbf{v}$

There are a lot of other properties we want these operations to have, but those aren't included among the axioms because they can be proved from them.

The set of axioms was selected so that (1) there are no redundancies in the sense that no one axiom can be proved from the rest, (2) everything we want can be proved from the axioms, and (3) each axiom is short and easy to understand.

There are alternative sets of axioms that could be used instead. For example, the first two axioms could be replaced by the single axiom $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{v}+(\mathbf{w}+\mathbf{u})$. In the presence of axiom 3, it's equivalent to the first two axioms taken together. There are a couple of reasons to prefer the two separate axioms 1 and 2. Axiom 1 only involves rearranging the terms, while axiom 2 doesn't change the order of the terms but reparenthesizes the expression. Thus, the two axioms address different concepts. The other reason for separating them is that there are mathematical operations that satisfy associativity but not commutativity. An example of that is composition of functions.

There's also a choice of which operations on vector spaces to axiomatize. Here we used addition and scalar multiplication. Alternatively, subtraction and scalar multiplication could be used. Addition was used because, in some sense, addition is a more basic operation than subtraction.

Properties that follow from the axioms. There are many of them. Here are some basic ones.

a. The sum of a finite list of vectors $\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k$ can be computed in any order, and fully parenthesized in any way, and the sum will be the same.

b. $\mathbf{v}+\mathbf{w}=\mathbf{0}$ if and only if $\mathbf{w}=-\mathbf{v}$.

c. The negation of $\mathbf{0}$ is $\mathbf{0}$: $-\mathbf{0}=\mathbf{0}$.

d. The negation of the negation of a vector is the vector itself: $-(-\mathbf{v})=\mathbf{v}$.

e. If $\mathbf{v}+\mathbf{z}=\mathbf{v}$, then $\mathbf{z}=\mathbf{0}$. Thus, $\mathbf{0}$ is the only vector that acts like $\mathbf{0}$.

f. Zero times any vector is the zero vector: $0\mathbf{v}=\mathbf{0}$ for every vector $\mathbf{v}$.

g. Any scalar times the zero vector is the zero vector: $c\mathbf{0}=\mathbf{0}$ for every real number c.

h. The only ways that the product of a scalar and an vector can equal the zero vector are when either the scalar is 0 or the vector is $\mathbf{0}$. That is, if $c\mathbf{v}=\mathbf{0}$, then either $c=0$ or $\mathbf{v}=\mathbf{0}$.

i. The scalar -1 times a vector is the negation of the vector: $(-1)\mathbf{v}=-\mathbf{v}$.

We define subtraction in terms of addition by defining $\mathbf{v}-\mathbf{w}$ as an abbreviation for $\mathbf{v}+(-\mathbf{w})$.

$$\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$$

All the usual properties of subtraction follow, such as

j. $\mathbf{u}+\mathbf{v}=\mathbf{w}$ if and only if $\mathbf{u}=\mathbf{w}-\mathbf{v}$.

k. $c(\mathbf{v}-\mathbf{w})=c\mathbf{v}-c\mathbf{w}$.

l. $(c-d)\mathbf{v}=c\mathbf{v}-d\mathbf{v}$

We'll prove some of these in class. We won't prove the first one, **a**. We'll look at some of the others instead.

The first one is important, however. It means we don't have to write parentheses when we're adding several vectors and we can rearrange the terms however we like. It follows from the commutativity and associativity axioms for addition. A formal proof would involve mathematical induction and takes time to formulate and construct.

Math 130 Home Page at
`http://math.clarku.edu/~ma130/`